Physicist / Engineer physicist BSc, year III / semester 1
"Optics problem solving" – practice notes

PRACTICE 9

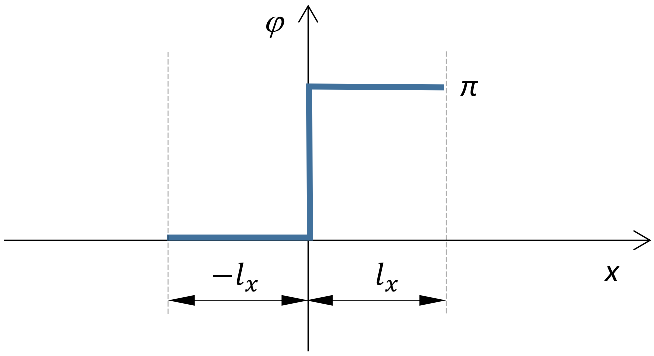
Dr. Gábor Erdei, 23-11-2025

1 Determining the far field of two slits in opposite phase by Fraunhofer diffraction

$$E(x',y') = \frac{e^{ikz'} \cdot e^{i\frac{k}{2z'}\left(x'^2+y'^2\right)}}{i\lambda z'} \iint_{\Sigma} E(x,y) \cdot e^{-i\frac{k}{z'}(x'x+y'y)} \mathrm{d}x\mathrm{d}y \tag{1}$$

Let the aperture be infinite in *y*-direction (one-dimensional analysis):

$$E(x') = \frac{e^{ikz'} \cdot e^{i\frac{kx'^2}{2z'}}}{\sqrt{i\lambda z'}} \int_{\Sigma} E(x) \cdot e^{-i\frac{kx'}{z'}x} \mathrm{d}x \tag{2}$$



The absolute value of field amplitude is unity inside an interval of $2l_x$ width, outside it is zero.

$$\int_{\Sigma} E(x) \cdot e^{-i\frac{kx'}{z'}x} \mathrm{d}x = \int_{-l_x}^{0} e^{-i\frac{kx'}{z'}x} \mathrm{d}x - \int_{0}^{l_x} e^{-i\frac{kx'}{z'}x} \mathrm{d}x =$$

$$= -\frac{z'}{ikx'}\left[e^{-i\frac{kx'}{z'}x}\right]_{-l_x}^{0} + \frac{z'}{ikx'}\left[e^{-i\frac{kx'}{z'}x}\right]_{0}^{l_x} = -\frac{z'}{ikx'}\left(2 - e^{i\frac{kx'}{z'}l_x} - e^{-i\frac{kx'}{z'}l_x}\right) =$$

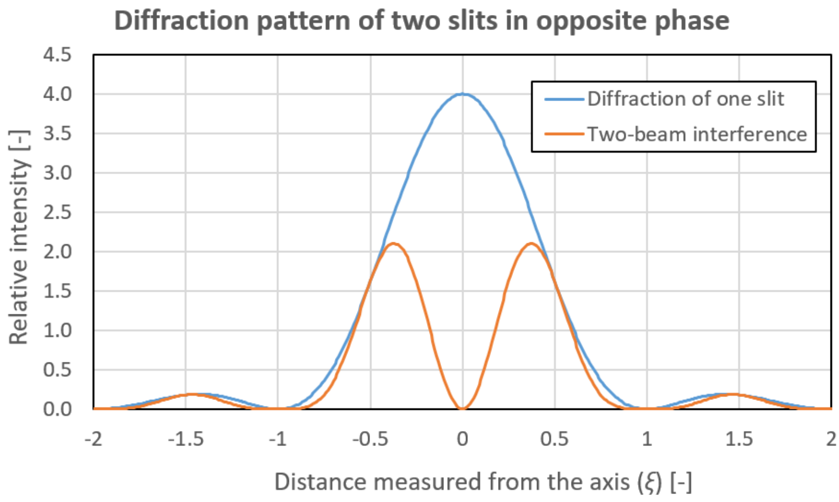
$$= -\frac{z'2}{ikx'}\left[1 - \cos\left(\frac{kx'}{z'}l_x\right)\right] = -\frac{z'2}{ikx'}\left[2\sin^2\left(\frac{kx'}{z'}\frac{l_x}{2}\right)\right] = i \cdot 2l_x \cdot \mathrm{sinc}\left(\frac{x'l_x}{\lambda z'}\right) \cdot \sin\left(\pi\frac{x'l_x}{\lambda z'}\right), \tag{3}$$

where we utilized that:

$$\cos 2\theta \equiv 1 - 2\sin^2\theta\,. \tag{4}$$

$$I(x') = \frac{v\varepsilon}{2}\frac{1}{\lambda z'} \cdot 4l_x^2 \cdot \mathrm{sinc}^2\left(\frac{x'l_x}{\lambda z'}\right) \cdot \sin^2\left(\pi\frac{x'l_x}{\lambda z'}\right). \tag{5}$$

$$\xi \triangleq \frac{x'l_x}{\lambda z'} \tag{6}$$



2 Curiosity: optical vortex

(source: wikipedia) The beam includes a 2π phase step, similarly to screw dislocations. Hence, the “wavefront” has a helical shape, with a phase singularity at its axis – all along which the field is zero. The beam is able to exert a torque on small particles. Application: e.g. in the astronomical charting of exoplanets it is used to suppress the light of the central star, or the Stimulated Emission Depletion (STED) microscopy.

3 Application of Fresnel diffraction: field distribution of a focused beam

$$E(x',y') = \frac{e^{ikz'} \cdot e^{i\frac{k\left(x'^2+y'^2\right)}{2z'}}}{i\lambda z'} \iint\limits_{\Sigma} E(x,y) \cdot e^{i\frac{k}{2z'}(x^2+y^2)} \cdot e^{-i\frac{k}{z'}(x'x+y'y)} \mathrm{d}x\mathrm{d}y \qquad (7)$$

see lecture notes…

Although it was a simplified derivation, we can get the same result analogously to the above from the Huygens-Fresnel integral too by a somewhat more complicated way (see Optical design course). Thus, the approximation is valid within the frame of this latter, i.e. if $NA < 0.5$.

4 E.g. field distribution of a rectangular aperture in the focal plane of a thin lens

$\rho = f$ (focal distance), $E(x, y) = E_0$

$$I(x',y') = \frac{\mathrm{v}\varepsilon}{2}\frac{E_0^2}{(\lambda f)^2}\, l_x^2 l_y^2 \,\mathrm{sinc}^2\left(\frac{x' l_x}{\lambda f}\right)\mathrm{sinc}^2\left(\frac{y' l_y}{\lambda f}\right) \qquad (8)$$

$$x_{\mathrm{Nyquist}} = \frac{\lambda f}{l_x} \qquad (9)$$

The same for a circular aperture:

$$R_{\mathrm{Airy}} = 1.22\frac{\lambda f}{D} \qquad (10)$$

E.g. a microscope objective of m = 20× magnification: NA = 0.4, f = 9.0 mm, λ = 550 nm.

$$\frac{D}{f} = 2\frac{D/2}{f} = 2\tan\theta \approx 2\sin\theta = 2NA \tag{11}$$

$$R_{\mathrm{Airy}} = 1.22\frac{\lambda}{2NA} = 0.61\frac{\lambda}{NA} = 0.61\frac{0.550\ \mu\mathrm{m}}{0.4} = 0.84\ \mu\mathrm{m} \tag{12}$$

5 Field distribution of a Gaussian beam in the focal plane of a thin lens (suppl.)

$\rho = f$ (focal distance). We assume that the lens is far more distant from the focal spot than the Rayleigh range ($f >> Z_R$), i.e. the center of the exit spherical wave is located at the beam waist indeed. We also assume that the lens aperture is much larger than the 1/e^2 radius (w) of the Gaussian beam. Then we can perform the integral from minus infinity to plus infinity.

$$E(x,y) \coloneqq e^{-\frac{r^2}{w^2}} = e^{-\frac{x^2+y^2}{w^2}} \tag{13}$$

$$E(x',y') = \frac{e^{i\frac{k}{2f}\left(x'^2+y'^2\right)}}{i\cdot\lambda f}\iint_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{w^2}}\cdot e^{-i\frac{k}{f}(x'x+y'y)}\mathrm{d}x\mathrm{d}y \tag{14}$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{w^2}}\cdot e^{-i\frac{kx'}{f}x}\mathrm{d}x = \int_{-\infty}^{\infty} e^{-\frac{x^2}{w^2}}\cdot\left(\cos\left(-\frac{kx'}{f}x\right) + i\sin\left(-\frac{kx'}{f}x\right)\right)\mathrm{d}x \tag{15}$$

The second term is antisymmetric, hence its integral is zero. That of the first term can be written in a closed form (see Abramowitz and Stegun (1972, p. 302, equation 7.4.6)):

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{w^2}}\cdot\cos\left(\frac{kx'}{f}x\right)\mathrm{d}x = \sqrt{\pi w^2}\cdot e^{-\pi^2 w^2\left(\frac{x'}{\lambda f}\right)^2} \tag{16}$$

Which is also a Gaussian function. The intensity:

$$I(x',y') = \left(\frac{\sqrt{\pi w^2}}{\lambda f}\right)^2\cdot e^{-2\pi^2 w^2\left(\frac{x'}{\lambda f}\right)^2}\cdot e^{-2\pi^2 w^2\left(\frac{y'}{\lambda f}\right)^2} = \frac{\pi w^2}{(\lambda f)^2}\cdot e^{-2\pi^2 w^2\left(\frac{r'}{\lambda f}\right)^2} \tag{17}$$

$$I(r') = \frac{\pi w^2}{(\lambda f)^2}\cdot e^{-2\left(\frac{r'}{\frac{\lambda f}{\pi w}}\right)^2} \tag{18}$$

So the 1/e^2 radius of the beam in focus (w_0, since the beam waist is here):

$$w_0 = \frac{\lambda f}{\pi w} = \frac{2}{\pi}\frac{\lambda f}{2w} = 0.637\frac{\lambda f}{2w}. \tag{19}$$

Both (18) and (19) are in perfect correspondence with what we learnt about the far-field propagation of Gaussian beams (see practice 2), with the following subsitution $z \rightarrow f$.