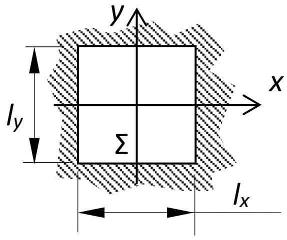
PRACTICE 8

Dr. Gábor Erdei, 19-11-2025

1 Determining the far-field light distribution of a rectangular aperture by Fraunhofer diffraction



$$E(x',y') = \frac{e^{ikz'} \cdot e^{i\frac{k}{2z'}\left(x'^2+y'^2\right)}}{i \cdot \lambda z'} \iint_{\Sigma} E(x,y) \cdot e^{-i\frac{k}{z'}(x'x+y'y)} \mathrm{d}x\mathrm{d}y \tag{1}$$

$$E(x,y) = \mathrm{rect}\left(\frac{x}{l_x}\right) \cdot \mathrm{rect}\left(\frac{y}{l_y}\right) \;\; ; \;\; E_0 \coloneqq 1 \;\; ; \;\; \mathrm{rect}(a) \triangleq \begin{cases} 1 \;\text{ if } |a| \leq \frac{1}{2} \\ 0 \;\text{ if } |a| > \frac{1}{2} \end{cases} \tag{2}$$

$$E(x',y') = \frac{e^{ikz'} \cdot e^{i\frac{k}{2z'}\left(x'^2+y'^2\right)}}{i \cdot \lambda z'} \int_{-l_y/2}^{l_y/2} \int_{-l_x/2}^{l_x/2} e^{-i\frac{k}{z'}(x'x+y'y)} \mathrm{d}x\mathrm{d}y \tag{3}$$

$$\int_{-l_x/2}^{l_x/2} e^{-i\frac{k}{z'}(x'x)} \mathrm{d}x = \left[-\frac{z'}{ikx'} e^{-i\frac{k}{z'}(x'x)}\right]_{-l_x/2}^{l_x/2} = -\frac{z'}{ikx'}\left[e^{-i\frac{k}{z'}\left(x'\frac{l_x}{2}\right)} - e^{+i\frac{k}{z'}\left(x'\frac{l_x}{2}\right)}\right] =$$

$$= -\frac{z'}{ikx'}\left[-2i \sin\frac{k}{z'}\left(x'\frac{l_x}{2}\right)\right] = \frac{2z'}{kx'}\left[\sin\frac{kx'}{2z'} l_x\right] = \frac{l_x}{l_x}\frac{\lambda z'}{\pi x'}\left[\sin\frac{\pi x' l_x}{\lambda z'}\right] = l_x \,\mathrm{sinc}\left(\frac{x' l_x}{\lambda z'}\right) \tag{4}$$

$$\mathrm{sinc}(\xi) \triangleq \frac{\sin \pi\xi}{\pi\xi} \;\; ; \;\; \xi \triangleq \frac{x' l_x}{\lambda z'} \tag{5}$$

$$E(x',y') = \frac{e^{ikz'} \cdot e^{i\frac{k}{2z'}\left(x'^2+y'^2\right)}}{i \cdot \lambda z'} \, l_x l_y \,\mathrm{sinc}\left(\frac{x' l_x}{\lambda z'}\right) \mathrm{sinc}\left(\frac{y' l_y}{\lambda z'}\right) \tag{6}$$

$$I(x',y') = \frac{\mathrm{v}\varepsilon}{2}\frac{1}{(\lambda z')^2} \, l_x^2 l_y^2 \,\mathrm{sinc}^2\left(\frac{x' l_x}{\lambda z'}\right) \mathrm{sinc}^2\left(\frac{y' l_y}{\lambda z'}\right) \tag{7}$$

Where is the first zero point?

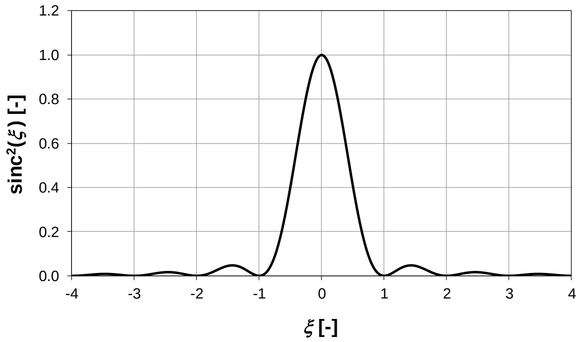
$$\pi\frac{x' l_x}{\lambda z'} = \pi \;\; \rightarrow \;\; \frac{x' l_x}{\lambda z'} = 1 \tag{8}$$

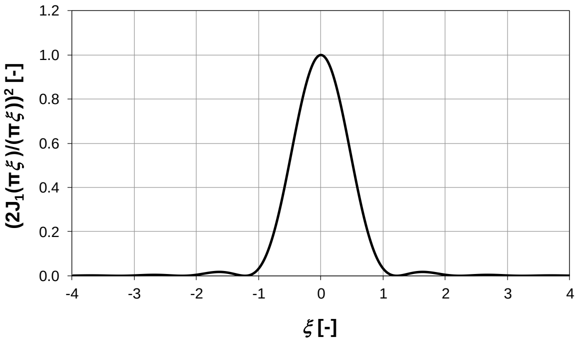
Nyquist aperture:

$$x_{\mathrm{Nyquist}} \triangleq x' = \frac{\lambda z'}{l_x} \tag{9}$$

In case of a circular aperture (Airy radius):

$$R_{\mathrm{Airy}} \triangleq r' = 1.22\frac{\lambda z'}{D} \;\; ; \;\; \xi \triangleq \frac{r' D}{\lambda z'} \tag{10}$$





E.g. What area of the objective lens (f = 30 mm) of a video projector is illuminated by one pixel of the spatial light modulator (SLM) of the projector (l_x = 7.6 μm), if λ = 550 nm? The lens is considered as ideal, placed at about a focal distance from the SLM.

$$x_{\text{Nyquist}} = \frac{\lambda z'}{l_x} = \frac{0.550 \cdot 30}{7.6} = 2.17 \text{ mm} \tag{11}$$

2 Resolving power of a telescope, see lecture notes

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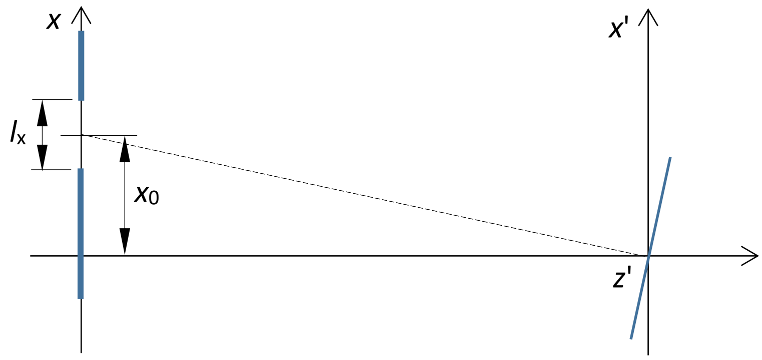
E.g. Very Large Telescope – VLT (ESO, Paranal Observatory), D = 8.2 m, λ = 550 nm

$$\Delta\theta' \approx \text{tg}\,\Delta\theta' = \frac{R_{\text{Airy}}}{z'} = 1.22\frac{\lambda}{D} = 8.2 \cdot 10^{-8} \text{ rad} \tag{12}$$

The angular field-of-view of the largest moon of Saturn, the Titan is: $3.9\cdot10^{-6}$ rad!

3 Far-field modelling of a rectangular aperture (slit) shifted from the axis



Let the value of the position shift be x_0: $E(x,y) \to E(x-x_0,y)$. According to (4):

$$\int_{x_0-l_x/2}^{x_0+l_x/2} e^{-i\frac{k}{z'}(x'x)}\mathrm{d}x = \left[-\frac{z'}{ikx'}e^{-i\frac{k}{z'}(x'x)}\right]_{x_0-l_x/2}^{x_0+l_x/2} =$$

$$= -\frac{z'}{ikx'}e^{-i\frac{k}{z'}(x'x_0)}\left[e^{-i\frac{k}{z'}\left(x'\frac{l_x}{2}\right)} - e^{+i\frac{k}{z'}\left(x'\frac{l_x}{2}\right)}\right] =$$

$$= e^{-i\frac{kx_0}{z'}x'} \cdot l_x \,\text{sinc}\left(\frac{x' l_x}{\lambda z'}\right) \tag{13}$$

In the obtained result a plane wave appears, which exactly corresponds to the shift theorem of Fourier transformation, if we thake the value of spatial frequncy to be $f_x = x'/(\lambda z')$. The shift of the slit does not cause any changes to the intensity distribution.

4 Far-field modelling of two rect. apertures shifted symmetrically from the axis

Let the shift be $+x_0$ and $-x_0$: : $E(x,y) \rightarrow E(x-x_0,y)+E(x+x_0,y)$. The calculation makes sense only when the slits do not overlap, i.e. $x_0 \geq l_x/2$. If $l_y >> l_x$, then the structure can be considered as one-dimensional. Based on (13):

$$\int_{x_0-l_x/2}^{x_0+l_x/2} e^{-i\frac{k}{z'}(x'x)}\mathrm{d}x + \int_{-x_0-l_x/2}^{-x_0+l_x/2} e^{-i\frac{k}{z'}(x'x)}\mathrm{d}x =$$

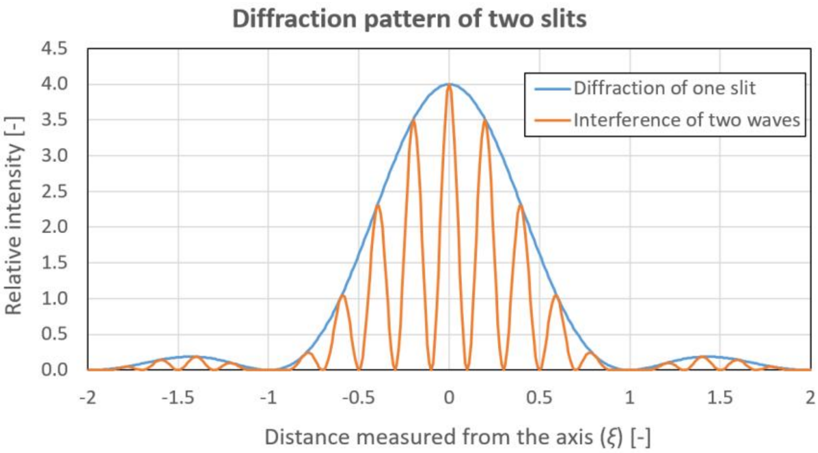
$$\left(e^{-i\frac{kx_0}{z'}x'}+e^{+i\frac{kx_0}{z'}x'}\right)\cdot l_x \operatorname{sinc}\left(\frac{x'l_x}{\lambda z'}\right) = 2\cos\left(\frac{kx_0}{z'}x'\right)\cdot l_x \operatorname{sinc}\left(\frac{x'l_x}{\lambda z'}\right) \tag{14}$$

So, adding a second slit does change the intensity distribution too:

$$I(x',y') = \frac{\mathrm{v}\varepsilon}{2}\frac{1}{(\lambda z')^2}\, l_x^2 l_y^2 \operatorname{sinc}^2\left(\frac{x'l_x}{\lambda z'}\right)\operatorname{sinc}^2\left(\frac{y'l_y}{\lambda z'}\right) 4\cos^2\left(\pi\frac{x'2x_0}{\lambda z'}\right) \tag{15}$$

Where $2x_0 = d$ is the distance between the slits. The above relationship is the scalar diffraction model of Young's two-slit experiment. According to $2\cos^2(a) \equiv 1+\cos(2a)$ we even get back the interferogram of two oblique plane waves (see lecture 7). The relationship is in perfect correspondence with the expression obtained at two-beam interference:

$$4\cos^2\left(\pi\frac{x'2x_0}{\lambda z'}\right) = 2\left(1+\cos\left(2\frac{2\pi}{\lambda}\frac{x_0}{z'}x'\right)\right) = 2(1+\cos(\Delta k' x')) \tag{16}$$



$d/l_x = 5$.

E.g. Let us place a single hair (of 50 µm thickness) in the middle of a narrow, long slit of 150 µm thickness, parallel with its longer side. The optical model of the arrangement is two slits of $l_x = 50$ µm width, with their centers placed d = 100 µm apart. Since $l_y >> l_x$ we make the calculation in one dimension, λ = 633 nm. What is the distance of the interference fringes on a screen being at a distance of z' = 200 mm?

$$2\frac{2\pi}{\lambda}\frac{x_0}{z'}x' = 2\pi \quad \rightarrow \quad x' = \frac{\lambda z'}{2x_0} = \frac{\lambda z'}{d} \tag{17}$$

$$x' = \frac{\lambda z'}{d} = \frac{0.633\ \mu\mathrm{m}\cdot 200\ \mathrm{mm}}{100\ \mu\mathrm{m}} = 1.27\ \mathrm{mm} \tag{18}$$

How many zeros does the interferogram have between the first zero points of the diffraction pattern of one slit (i.e. inside the two-times Nyquist aperture)?

$$2x_{\mathrm{Nyquist}} = 2\frac{\lambda z'}{l_x} \tag{19}$$

$$\frac{2x_{\mathrm{Nyquist}}}{x'} = \frac{2d}{l_x} = 4\ \mathrm{pcs.} \tag{20}$$