Physicist / Engineer physicist BSc, year III / semester 1
"Optics problem solving" – practice notes

PRACTICE 7

Dr. Gábor Erdei, 12-11-2025

1 Application of multiple-beam interferometry in case of diffraction gratings

f_y = 1200 lp/mm, D = 50 mm, scalar approximation, normal incidence, λ = 500 nm. What is the angle of the first diffraction order, $\theta_{m=1}$? Grating period $a = 1/f_y$ = 833.3 nm. Grating equation:

$$\delta = \frac{2\pi}{\lambda}\sin\theta_m \cdot a = m2\pi \tag{1}$$

If m = 1:

$$\sin\theta_{m=1} = \frac{\lambda}{a} = 0.6 \quad\rightarrow\quad \theta_{m=1} = 36.9°. \tag{2}$$

Relative to this, how much degrees of $\Delta\theta$ angle deviation is the first zero position? The direction characteristics of the diffracted light:

$$I(\delta) = I_1\left(\frac{\sin(\delta N/2)}{\sin(\delta/2)}\right)^2, \tag{3}$$

its first zero is located here:

$$\Delta\delta N/2 = \pi \quad\rightarrow\quad \Delta\delta = \frac{2\pi}{N} \tag{4}$$

From (1):

$$\Delta\delta = \frac{2\pi}{\lambda}\Delta\sin\theta \cdot a = \frac{2\pi}{N} \quad\rightarrow\quad \Delta\sin\theta = \frac{\lambda}{aN} = \frac{\lambda}{D} \tag{5}$$

Using that $N \gg 1$, i.e. $\Delta\delta \ll 2\pi$, from (5) we can easily calculate $\Delta\theta$:

$$\Delta\sin\theta \approx \frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta}\Delta\theta = \Delta\theta\cdot\cos\theta = \frac{\lambda}{aN} = \frac{\lambda}{D} \tag{6}$$

$$\Delta\theta\cdot 0.8 = 10^{-5} \quad\rightarrow\quad \Delta\theta = 0.013\ \mathrm{mrad}. \tag{7}$$

By what $\Delta\lambda$ value shall we detune the wavelength so that the direction of the first diffraction order changes exactly by the above-determined value (Rayleigh criterion)? Using (2) and (5):

$$\sin\theta_{m=1} + \Delta\sin\theta = \frac{\lambda+\Delta\lambda}{a} \tag{8}$$

$$\frac{\lambda}{a} + \frac{\lambda}{aN} = \frac{\lambda+\Delta\lambda}{a} \quad\rightarrow\quad \Delta\lambda = \frac{\lambda}{N} \tag{9}$$

Since $N = D/a = 60\,000$:

$$\Delta\lambda = 8.33\ \mathrm{pm}. \tag{10}$$

The above value can be obtained directly from the formula of resolving power $\mathcal{R}$ (see lecture):

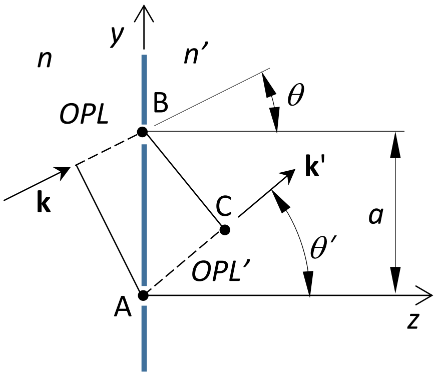
$$\mathcal{R} \triangleq \frac{\lambda}{\Delta\lambda} = m\cdot N. \tag{11}$$

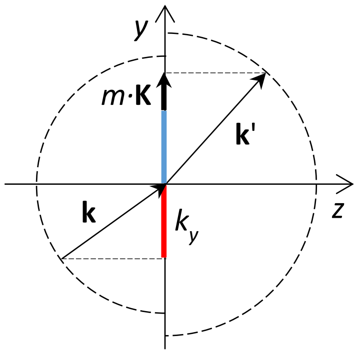
2 Transmissive diffraction grating in case of oblique incidence

At the slit in point "A" the phase of a plane wave of θ angle of incidence is: φ_A, and in "B" it is

$$\varphi_\mathrm{B} = \varphi_\mathrm{A} + \frac{2\pi}{\lambda_0}OPL. \tag{12}$$

Looking at θ' angle in the far field, the wave diffracted by a slit in point "B" is observed as a plane wave going in this direction. We virtually trace this plane wave back to the grating, and draw its wavefront crossing point “B”, over which the phase is constant with a value as determined by (12). We compare the phases of waves diffracted from slits “A” and “B” along this wavefront in point “C”. Here the phase of the wave diffracted by slit “A” is:





$$\varphi_{\mathrm{C}} = \varphi_{\mathrm{A}} + \frac{2\pi}{\lambda_0} OPL'. \tag{13}$$

The difference in the phases obtained above provides the phase difference between the plane wave components of multiple beam interference, being constructive on the below condition:

$$\delta = \varphi_{\mathrm{C}} - \varphi_{\mathrm{B}} = \frac{2\pi}{\lambda_0} OPL' - \frac{2\pi}{\lambda_0} OPL = m2\pi \tag{14}$$

$$OPL = n \sin\theta \cdot a \quad ; \quad OPL' = n' \sin\theta' \cdot a \tag{15}$$

$$n' \frac{2\pi}{\lambda_0} \sin\theta' \cdot a - n \frac{2\pi}{\lambda_0} \sin\theta \cdot a = m2\pi \tag{16}$$

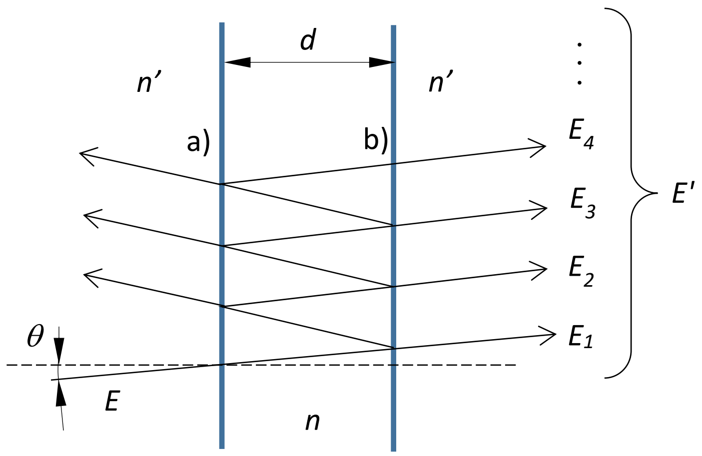
$$n' \frac{2\pi}{\lambda_0} \sin\theta' = n \frac{2\pi}{\lambda_0} \sin\theta + \frac{m2\pi}{a} \tag{17}$$

$$k'_y = k_y + m\frac{2\pi}{a} = k_y + mK \tag{18}$$

$$\mathbf{K} \triangleq \hat{\mathbf{y}} \cdot \frac{2\pi}{a}, \tag{19}$$

where **K** is called the “grating vector”. Equation (18) exhibits a perfect analogy with phase matching studied at Fresnel reflection. An interesting fact is that since the momentum of photons is $\hbar\mathbf{k}$, in case of diffraction gratings the law of momentum conservation is not satisfied in *y* direction, since the grating interacts with the incident beam.

3 Fabry-Pérot interferometer



The phase difference (in case of normal incidence) for one back and forth reflection is:

$$\delta = \frac{2\pi}{\lambda_0} 2dn \tag{20}$$

Normal incidence ($\theta \approx 0°$), the complex amplitude for the incident field is E, for the transmitted field it is E'. Since the transmission coefficient on the two surfaces will not be uniform even for identical mirrors if $n \neq n'$ (only the transmittance T is the same), thus we denote them individually with τ_a and τ_b:

$$E_1 = E \cdot \tau_\mathrm{a} \cdot e^{i\delta/2} \cdot \tau_b \tag{21}$$

$$E_2 = E \cdot \tau_\mathrm{a} \cdot e^{i\delta/2} \cdot \rho \cdot e^{i\delta} \cdot \rho \cdot \tau_\mathrm{b} \tag{22}$$

$$E_3 = E \cdot \tau_\mathrm{a} \cdot e^{i\delta/2} \cdot \rho^2 \cdot e^{i\delta} \cdot \rho^2 \cdot e^{i\delta} \cdot \tau_\mathrm{b} \tag{23}$$

$$E_N = E \cdot \tau_\mathrm{a}\tau_\mathrm{b} \cdot e^{i\delta/2} \cdot \left(\rho^2 \cdot e^{i\delta}\right)^{N-1} \tag{24}$$

If $|q| < 1$, then

$$S_N = a_1 \frac{1 - q^N}{1 - q} \quad ; \quad \lim_{N\to\infty} S_N = a_1 \frac{1}{1 - q} \quad ; \quad a_1 = E \cdot \tau_\mathrm{a}\tau_\mathrm{b} \cdot e^{i\delta/2} \quad ; \quad q = \rho^2 \cdot e^{i\delta} \tag{25}$$

$$E' = \sum_{j=1}^{N\to\infty} E_j = E \cdot \tau_\mathrm{a}\tau_\mathrm{b} \cdot e^{i\frac{\delta}{2}} \cdot \frac{1}{1 - \rho^2 \cdot e^{i\delta}} \tag{26}$$

Intensity calculated from the Poynting vector definition (here the outside media are identical):

$$I = \frac{\mathrm{v}\varepsilon}{2} |E|^2 \quad \text{and} \quad I' = \frac{\mathrm{v}\varepsilon}{2} |E'|^2, \tag{27}$$

as well as we learned at the Fresnel formulae that in case of normal incidence:

$$T = |\tau_\mathrm{a}|^2 \cdot \frac{n}{n'} = |\tau_\mathrm{b}|^2 \cdot \frac{n'}{n} \;\Rightarrow\; |\tau_\mathrm{a}|^2 \cdot |\tau_\mathrm{b}|^2 = T \cdot \frac{n'}{n} \cdot T \cdot \frac{n}{n'} = T^2 \tag{28}$$

By multiplication with the complex conjugate we get the following from (26) (when ρ is real):

$$I' = I \cdot T^2 \cdot \frac{1}{1 - R \cdot e^{-i\delta} - R \cdot e^{i\delta} + R^2} = \frac{I \cdot T^2}{1 - R \cdot 2\cos\delta + R^2} \tag{29}$$

Using that $\cos 2\alpha \equiv 1 - 2\sin^2\alpha$:

$$\frac{I'}{I} = \frac{T^2}{1 - 2R(1 - 2\sin^2(\delta/2)) + R^2} = \frac{T^2}{(1-R)^2 + 4R\sin^2(\delta/2)} = \left(\frac{T}{1-R}\right)^2 \frac{1}{1 + \frac{4R}{(1-R)^2}\sin^2(\delta/2)} \tag{30}$$

If there is no absorption in the layers, then $R + T = 1$. Airy function:

$$\frac{I'}{I} = \frac{1}{1 + \frac{4R}{(1-R)^2}\sin^2(\delta/2)}. \tag{31}$$

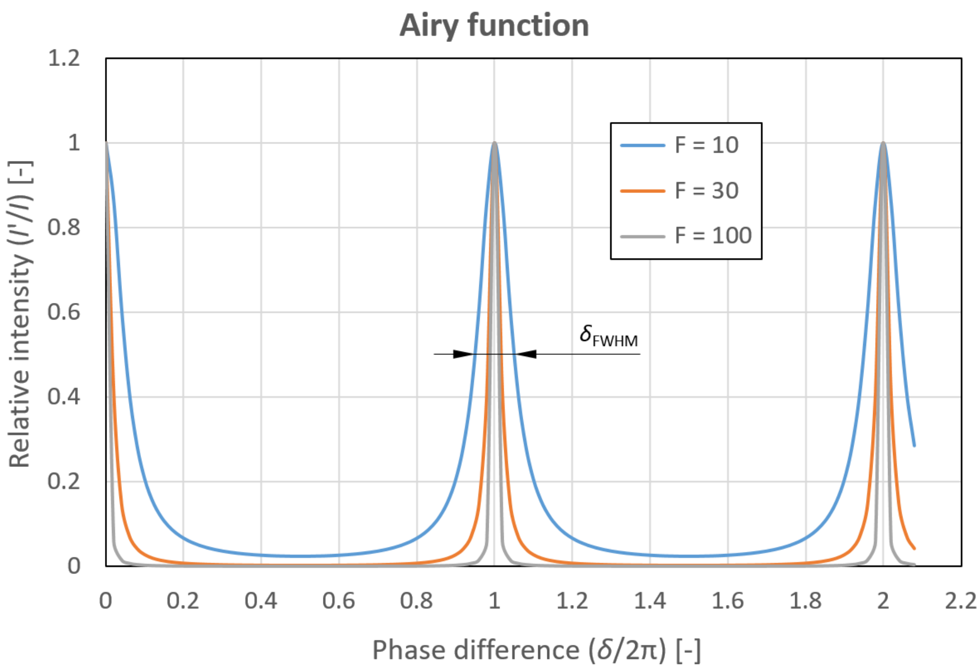
Positions of interference peaks:

$$\delta/2 = m \cdot \pi \quad \to \quad \delta = m \cdot 2\pi \qquad m \in \mathbb{N}^+ \tag{32}$$

If ρ is not real, then the phase shift caused by the complex reflection coefficient adds to δ, laterally shifting the Airy function. In case of a potential absorption in the mirror layers the peaks will not reach 1. The full-width-at-half maximum of the peaks is:

$$\frac{1}{1 + \frac{4R}{(1-R)^2}\sin^2\left(\delta_{1/2}/2\right)} = \frac{1}{2} \tag{33}$$

$$\sin\frac{\delta_{1/2}}{2} = \frac{1 - R}{2\sqrt{R}} \tag{34}$$



If R is close to unity, then we get a very small number on the right, i.e.

$$\sin\frac{\delta_{1/2}}{2} \approx \frac{\delta_{1/2}}{2} \tag{35}$$

$$\delta_{1/2} = \frac{1-R}{\sqrt{R}} \quad\rightarrow\quad \delta_{\text{FWHM}} = 2\frac{1-R}{\sqrt{R}} \tag{36}$$

$$\mathcal{F} \triangleq \frac{2\pi}{\delta_{\text{FWHM}}} = \frac{\pi\sqrt{R}}{1-R}. \tag{37}$$

This is called the "finesse", its typical value lies between 30..1000. It can be easily proven that if $R \rightarrow 1.0$, then the Airy function tends to the so-called Lorentz distribution.

4 How much shall the wavelength be tuned so that the phase changes exactly by δ_{FWHM}?

$$\frac{2\pi}{\lambda+\Delta\lambda}2d - \frac{2\pi}{\lambda}2d = \delta_{\text{FWHM}} \tag{38}$$

Since

$$\frac{1}{1+\mathrm{b}} \approx 1-b\,,\quad \text{if}\quad b \ll 1 \tag{39}$$

$$\frac{1}{\lambda+\Delta\lambda} = \frac{1}{\lambda}\frac{1}{1+\frac{\Delta\lambda}{\lambda}} \approx \frac{1}{\lambda}\left(1-\frac{\Delta\lambda}{\lambda}\right) = \frac{1}{\lambda} - \frac{\Delta\lambda}{\lambda^2} \tag{40}$$

Writing this into (38):

$$-\frac{\Delta\lambda}{\lambda^2}2\pi 2d = \delta_{\text{FWHM}} \tag{41}$$

The value of the resolving power $\mathcal{R}$ (Taylor criterion), can be obtained by rearranging (41):

$$\mathcal{R} \triangleq \left|\frac{\lambda}{\Delta\lambda}\right| = \frac{1}{\delta_{\text{FWHM}}}\frac{2\pi 2d}{\lambda} = \frac{m\cdot 2\pi}{\delta_{\text{FWHM}}} = m\cdot\mathcal{F}. \tag{42}$$

Here *m* can be much larger than at diffraction gratings! The interference peaks are called as longitudinal resonator modes in laser technology. E.g. He-Ne laser: λ = 633 nm, in case of *d* = 150 mm:

$$m = \frac{2d}{\lambda} = \frac{300 \text{ mm}}{633 \cdot 10^{-6} \text{ mm}} = 473\,934\,! \tag{43}$$

The reflectance of the end mirror of a He-Ne laser is higher than 99.9%, that of the out coupler mirror it is about 99%. If we substitute their geometrical average into the formula for finesse:

$$\mathcal{F} = \frac{\pi\sqrt{R}}{1-R} = \frac{\pi\sqrt{0.9945}}{1-0.9945} = 569 \tag{44}$$

From this the resolving power

$$\mathcal{R} = m \cdot \mathcal{F} = 2.695 \cdot 10^{8} \tag{45}$$

According to (42) the full-width-at-half maximum of an interference peak is:

$$\Delta\lambda = \frac{\lambda}{\mathcal{R}} = \frac{633 \text{ nm}}{2.695 \cdot 10^{8}} = 0.00235 \text{ pm}\,! \tag{46}$$

In frequency this corresponds to a bandwidth of 1.76 MHz (see e.g. single-mode lasers). Ordinary He-Ne lasers operate in multiple modes simultaneously (typically 2-5), which are separated by 200-1000 MHz from each other.

Applications: laser resonators, Fabry-Pérot etalons, optical band-pass filters, spectrum analyzers, cavity ring-down spectroscopy etc.