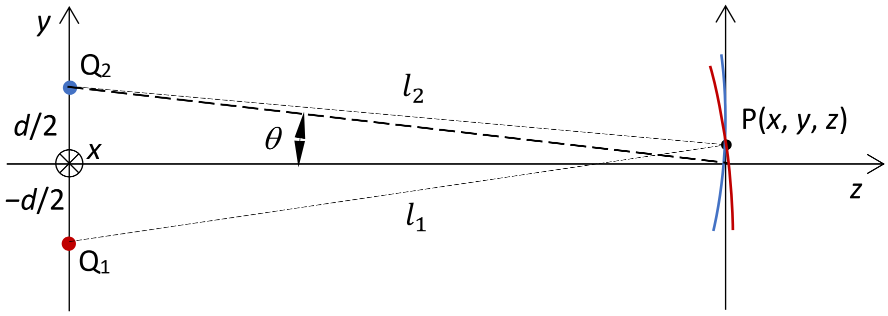
PRACTICE 6

Dr. Gábor Erdei, 05-11-2025

1 Interference of two spherical waves in paraxial approximation

Polarization: *x*-direction, scalar approximation.



$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta \tag{1}$$

$$E_{1,2} \coloneqq \frac{E_0}{l_{1,2}} e^{ikl_{1,2}} \;\; ; \;\; k = \frac{2\pi}{\lambda} \tag{2}$$

$$\delta = kl_1 - kl_2 \tag{3}$$

$$l_1 = \sqrt{z^2 + \left(y + \frac{d}{2}\right)^2 + x^2} \quad \text{and} \quad l_2 = \sqrt{z^2 + \left(y - \frac{d}{2}\right)^2 + x^2} \tag{4}$$

Assumptions: *d*, *y*, *x* << *z*. For this reason in the denominator of (2) $l_{1,2} \approx z$, i.e. $I_1 \approx I_2$, and:

$$I = 2I_1(1 + \cos\delta). \tag{5}$$

Determining the phase:

$$l_1 = z\sqrt{1 + \left(\frac{y + \frac{d}{2}}{z}\right)^2 + \frac{x^2}{z^2}} \approx z\left(1 + \frac{1}{2}\left[\left(\frac{y + \frac{d}{2}}{z}\right)^2 + \frac{x^2}{z^2}\right]\right) = z + \frac{\left(y + \frac{d}{2}\right)^2}{2z} + \frac{x^2}{2z} \tag{6}$$

$$l_2 \approx z + \frac{\left(y - \frac{d}{2}\right)^2}{2z} + \frac{x^2}{2z} \tag{7}$$

$$\delta = k\left(\frac{\left(y + \frac{d}{2}\right)^2}{2z} - \frac{\left(y - \frac{d}{2}\right)^2}{2z}\right) = \frac{2\pi}{\lambda}\frac{2yd}{2z} = \frac{2\pi}{\lambda}\frac{yd}{z}. \tag{8}$$

Conclusion: there is no *x*-dependence! Shape of interference fringes (δ = const.):

$$\delta = \frac{2\pi}{\lambda}\frac{yd}{z} \coloneqq m \cdot 2\pi \;\; ; \;\; m \in \mathbb{Z} \tag{9}$$

$$y = m\lambda\frac{z}{d} = \frac{m\lambda}{2 \cdot \tan\theta} \approx \frac{m\lambda}{2\theta}. \tag{10}$$

If at y = 50 mm we see m = 50 fringes, then by (10), for $\lambda = 633$ nm, z/d = 1580! Thus we can use approximation $d \ll z$, indeed. Application: e.g. Young's two-slit experiment (wavefront splitting), shearing interferometer (amplitude splitting).

2 Analysis of a Fizeau interferometer with two-beam interference

First let us examine the behavior of a plane wave reflected by a plane-parallel plate at close-perpendicular incidence! The EM wave reflected by a plane-parallel plate of n refractive index and d thickness can be approximated by two-beam interference, when the surface reflectance is small. The phase difference in case of an incident plane-wave (see lecture):

$$\delta = k_0 \cdot 2nd \cdot \cos\theta_2 + \pi \tag{11}$$

$$\cos\theta_2 \approx 1 - \frac{\theta_2^2}{2} + \cdots \qquad [\theta_2] = \mathrm{rad} \tag{12}$$

If we consider close-perpendicular incidence, then

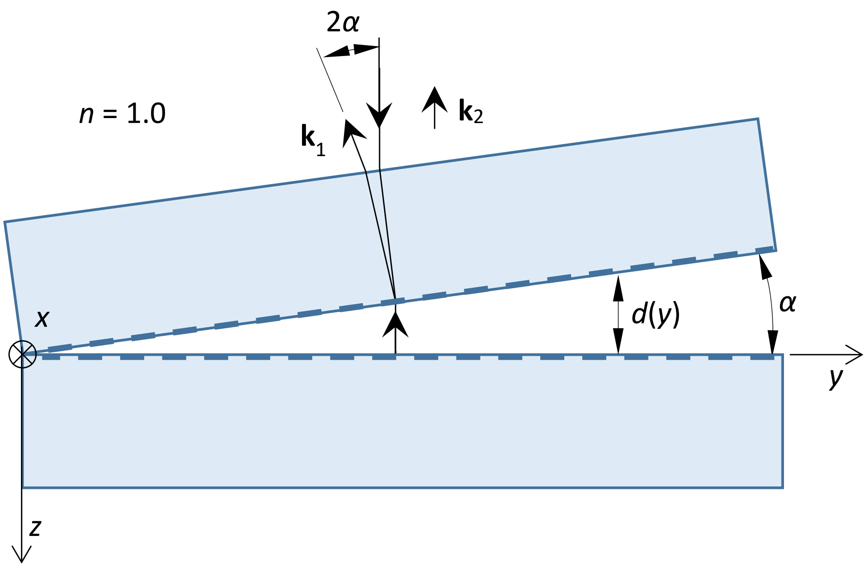
$$\frac{\theta_2^2}{2} \ll 1 \;\; \rightarrow \;\; |\theta_2| \ll \sqrt{2}\,\mathrm{rad} = \; 81° \;\; \rightarrow \;\; \theta_2 < 8° \quad (\cos 8° = 0.990) \tag{13}$$

that is $\cos\theta_2 \approx 1$. With this:

$$\delta \approx k_0 \cdot 2nd + \pi, \tag{14}$$

that is, we can calculate with a very good precision as if the angle of incidence were zero.

Now let us have a look at the arrangement in the below figure. Let n = 1.0. i.e. air, as well as let the touching surfaces of the two plane-parallel glass plates indicated by dashed lines serve as the bottom and top interfaces participating in two-beam interference. Let us consider as if the other surfaces of the glass plates were covered by anti-reflection coatings, thus we can neglect their reflections. If the thickness of the air gap between the plates is not constant, but slightly wedged (i.e. changes linearly), then we get a so-called Fizeau interferometer.



The plane waves reflected by the two surfaces will not be perfectly parallel with each other. During the lecture, we analyzed this case too:

$$\delta = (\mathbf{k}_2 - \mathbf{k}_1)\mathbf{r} + \Delta\varphi = \Delta\mathbf{k} \cdot \mathbf{r} + \Delta\varphi \tag{15}$$

Both interfering plane waves traverse the top plate, which causes a constant phase shift (independent of y) in consequence, so we can neglect it from our analysis. We can also imagine as if its thickness were almost zero. The difference vector always lies in the y-z plane, thus there is no x-dependence here either:

$$\Delta\mathbf{k} = k_0\big((\sin 2\alpha)\cdot\hat{\mathbf{y}} + (-1+\cos 2\alpha)\cdot\hat{\mathbf{z}}\big), \tag{16}$$

If α is sufficiently small, then $\Delta\mathbf{k}$ vector is almost parallel to the y direction (i.e. $\Delta\mathbf{k}_z \approx 0$). Hence, examining the error is perfectly analogous with (13), only here

$$2\alpha < 8° \text{ that is } \alpha < 4° \tag{17}$$

$$\Delta\mathbf{k} \approx k_0(\sin 2\alpha)\cdot\hat{\mathbf{y}} \approx k_0 2\alpha\cdot\hat{\mathbf{y}} \quad [\alpha] = \mathrm{rad} \tag{18}$$

If $\mathbf{r} = y\cdot\hat{\mathbf{y}}$, then

$$\delta = k_0 2\alpha\cdot y + \Delta\varphi \tag{19}$$

Since the optical path difference is zero where $y = 0$, thus $\Delta\varphi = \pi$, correspondingly:

$$\delta = k_0 2\alpha\cdot y + \pi \tag{20}$$

The position of the interference fringes:

$$\delta = \frac{2\pi}{\lambda_0}\cdot 2\alpha\cdot y + \pi = m\cdot 2\pi \ ; \ \ m \in \mathbb{N}^+ \tag{21}$$

$$y = \frac{m\cdot 2\pi - \pi}{2\alpha}\frac{\lambda_0}{2\pi} = \frac{2m-1}{4\alpha}\lambda_0 \tag{22}$$

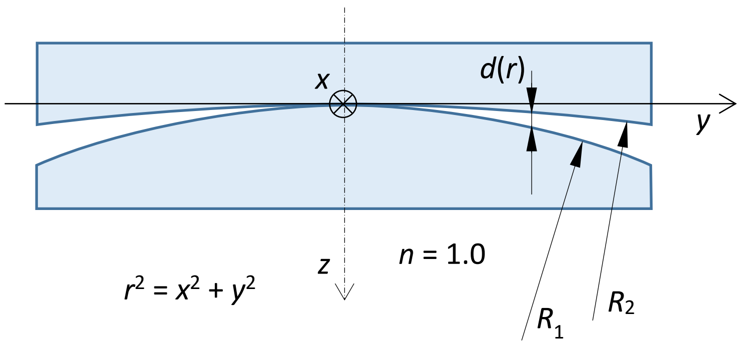
Note the wavelength dependence of the fringe position! Assuming that we see m = 50 fringes up to y = 50 mm, then by (22), for $\lambda_0 = 633$ nm we get $\alpha = 0.018°$, which is zero to a good approximation, indeed! Since 2α is very small, the next relationship can also be observed:

$$\alpha \approx \frac{d(y)}{y} \tag{23}$$

$$\delta(y) = \frac{2\pi}{\lambda_0}\cdot 2d(y) + \pi. \tag{24}$$

Application: typically surface shape measurement.

3 Analysis of Newton fringes with two-beam interference



$$R_{1,2}^2 = r^2 + \left(z - R_{1,2}\right)^2 \ \rightarrow \ z = R_{1,2} - \sqrt{R_{1,2}^2 - r^2} \tag{25}$$

From the ± signs needed after a root we opted for the negative, since this describes the upper hemisphere of the complete sphere crossing the origo. To satisfy (13) we need r to be much smaller than $R_{1,2}$. Then we can approximate (25) by the first two terms of its Taylor-series:

$$z \approx \frac{r^2}{2R_{1,2}}, \tag{26}$$

$$d(r) = \frac{r^2}{2R_1} - \frac{r^2}{2R_2} = \frac{r^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \tag{27}$$

In order to fulfill condition (17) the (local) angle between the two surfaces must be small:

$$\alpha \approx \frac{\partial d}{\partial r} = r\left(\frac{1}{R_1} - \frac{1}{R_2}\right) < \left(4^\circ \cdot \frac{\pi}{180^\circ}\right) = 0.07\ [\mathrm{rad}] \tag{28}$$

Since $r << R_{1,2}$ this is satisfied right away. Let us consider the two most important special cases:

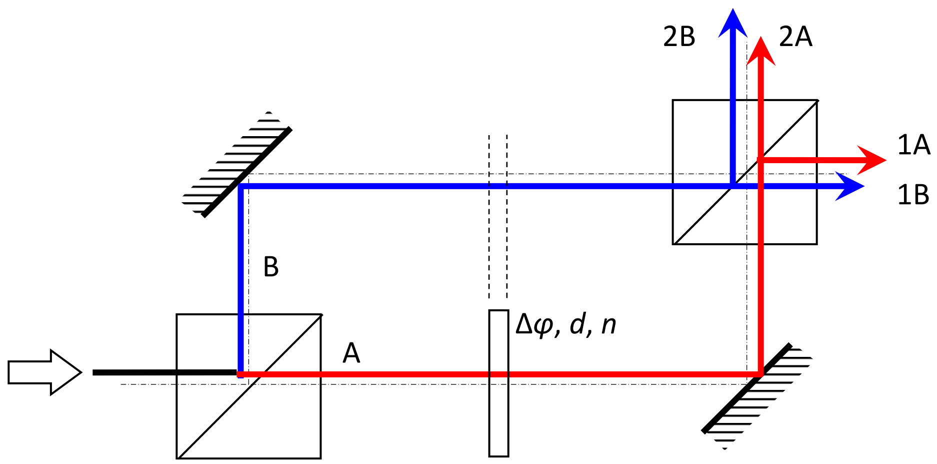
a) If $R_1 \approx R_2$ and $R_2 \coloneqq R_1 + \Delta R$. Application: test glass for checking optical surfaces.

$$d(r) = \frac{r^2}{2}\left(\frac{R_2 - R_1}{R_1 R_2}\right) \approx \frac{r^2}{2}\frac{\Delta R}{R_1^2} \quad ; \quad \delta(r) = \frac{2\pi}{\lambda_0} \cdot \frac{r^2}{R_1^2}\Delta R + \pi \tag{29}$$

b) If $1/R_2 = 0$ we obtain the classic Newton fringes ($m \in \mathbb{N}^+$):

$$d(r) = \frac{r^2}{2}\frac{1}{R_1} \quad ; \quad \delta(r) = \frac{2\pi}{\lambda_0} \cdot \frac{r^2}{2R_1} + \pi = m \cdot 2\pi \quad \Rightarrow \quad r = \sqrt{(2m-1)R_1\lambda_0} \tag{30}$$

4 Analysis of a Mach-Zehnder interferometer with two-beam interference



For the φ phase shift let us calculate what the optical path difference (*OPD*) is inside the glass plate and in the other arm of the interferometer in air, after travelling a *d* distance:

$$OPD = n \cdot d - 1.0 \cdot d = (n - 1.0) \cdot d \tag{31}$$

$$\Delta\varphi = k_0 OPD = \frac{2\pi}{\lambda_0}(n - 1.0)d \tag{32}$$

$$\delta_1 = \Phi_{1\mathrm{A}} - \Phi_{1\mathrm{B}} = \left(0 + \Delta\varphi + \frac{\pi}{2}\right) - \left(\frac{\pi}{2} + 0\right) = \frac{2\pi}{\lambda_0}(n-1)d \tag{33}$$

$$\delta_2 = \Phi_{2\mathrm{A}} - \Phi_{2\mathrm{B}} = (0 + \Delta\varphi + 0) - \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{2\pi}{\lambda_0}(n-1)d - \pi \tag{34}$$

This is a single-pass interferometer! Application: measuring the refractive index of gases, optical signal processing, data storage.