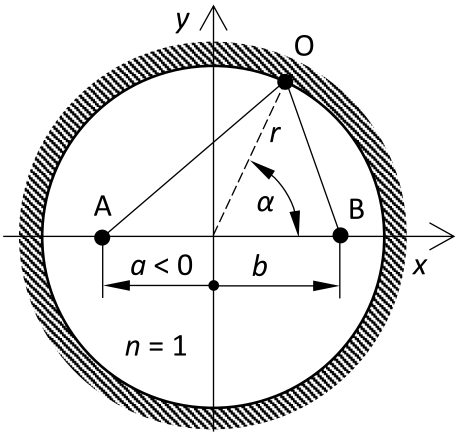
PRACTICE 5

Dr. Gábor Erdei, 15-10-2025

1 Finishing up the paraxial material left out from the last practice…

2 Application of Fermat's principle to determine ray trajectory

Which α value makes the AOB segment a real ray in the spherical mirror of the below figure?



$$\begin{aligned} x_O &= r\cos(\alpha) \\ y_O &= r\sin(\alpha) \end{aligned} \tag{1}$$

$$\mathrm{AO} = \sqrt{r^2\sin^2(\alpha) + (r\cos(\alpha) - a)^2} \tag{2}$$

$$\mathrm{OB} = \sqrt{r^2\sin^2(\alpha) + (b - r\cos(\alpha))^2} \tag{3}$$

$$OPL = \mathrm{AO} + \mathrm{OB} \tag{4}$$

$$\mathrm{AO} = \sqrt{r^2\sin^2(\alpha) + r^2\cos^2(\alpha) - 2ar\cos(\alpha) + a^2} = \sqrt{r^2 - 2ar\cos(\alpha) + a^2} \tag{5}$$

$$\mathrm{OB} = \sqrt{r^2\sin^2(\alpha) + r^2\cos^2(\alpha) - 2br\cos(\alpha) + b^2} = \sqrt{r^2 - 2br\cos(\alpha) + b^2} \tag{6}$$

$$\frac{\mathrm{d}OPL}{d\alpha} = 0 \tag{7}$$

$$\frac{1}{2}\frac{2ar\sin(\alpha)}{\sqrt{r^2 - 2ar\cos(\alpha) + a^2}} + \frac{1}{2}\frac{2br\sin(\alpha)}{\sqrt{r^2 - 2br\cos(\alpha) + b^2}} = 0 \tag{8}$$

Here we can simplify with $\sin(\alpha)$, if its value is non-zero. This occurs when $\alpha = 0°$, which happens to be the trivial solution to the problem. (In this case light travels along the *x* axis.)

$$\frac{a}{\sqrt{r^2 - 2ar\cos(\alpha) + a^2}} = -\frac{b}{\sqrt{r^2 - 2br\cos(\alpha) + b^2}} \tag{9}$$

$$a^2(r^2 - 2br\cos(\alpha) + b^2) = b^2(r^2 - 2ar\cos(\alpha) + a^2) \tag{10}$$

$$a^2r - 2a^2b\cos(\alpha) = b^2r - 2b^2a\cos(\alpha) \tag{11}$$

$$\cos(\alpha) = \frac{b^2r - a^2r}{2b^2a - 2a^2b} = \frac{r}{2}\frac{b^2 - a^2}{ab(b-a)} = \frac{r}{2}\frac{b+a}{ab}. \tag{12}$$

3 Complex refractive index application example: absorption/reflection of gold

λ_0 = 400 nm, normal incidence from air onto gold surface, with linear (x, i.e. p) polarization. n = 1.468, κ = 1.953. What is the surface reflectance? We consider the refractive index of air to be 1.0. What portion of the incident intensity (I_{in}) can go through a gold layer of d = 20 nm thickness? (Multiple reflections can be neglected.)

$$\tilde{n} \triangleq n - \mathrm{i} \cdot \kappa \tag{13}$$

$$R = \left|\frac{1-\tilde{n}}{1+\tilde{n}}\right|^2 = \frac{1-1.468+\mathrm{i}\cdot 1.953}{1+1.468-\mathrm{i}\cdot 1.953} \cdot \frac{1-1.468-\mathrm{i}\cdot 1.953}{1+1.468+\mathrm{i}\cdot 1.953} = \frac{0.468^2+1.953^2}{2.468^2+1.953^2} = 40.7\%. \tag{14}$$

How much is the penetration depth? Axis z points in the direction of the surface normal.

$$E_x(z) = E_{0x} \cdot e^{-ik_{\mathrm{re}}z} \cdot e^{-k_{\mathrm{im}}z} \tag{15}$$

Intensity attenuation is described by the Lambert-Beer law, i.e. the absolute square of (15):

$$I(z) = I_0 \cdot e^{-2k_{\mathrm{im}}z}, \tag{16}$$

where I_0 is the intensity in gold at z = 0 position. For normal incidence it is true that:

$$k_{\mathrm{im}} = k_0 \cdot \kappa \tag{17}$$

$$I(z) = I_0 \cdot e^{-2k_0\cdot\kappa\cdot z} = I_0 \cdot e^{-2\frac{z}{\delta}} \tag{18}$$

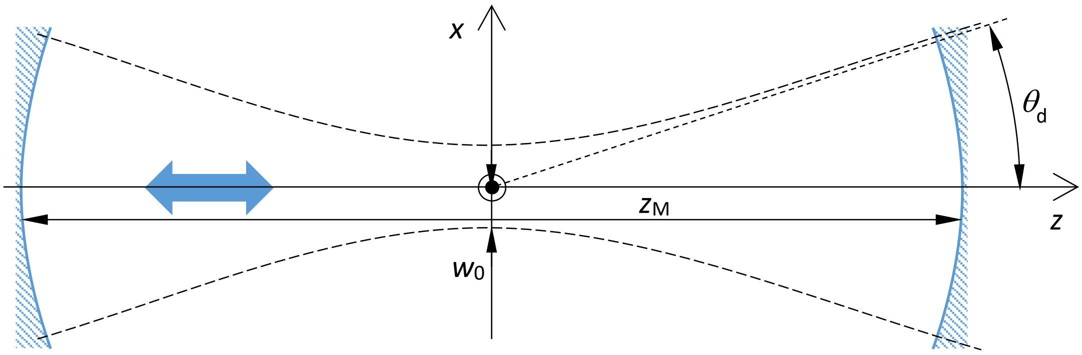
$$\delta = \frac{1}{k_0 \cdot \kappa} = \frac{\lambda_0}{2\pi \cdot \kappa} = \frac{400\ \mathrm{nm}}{2\pi \cdot 1.953} = 32.6\ \mathrm{nm}. \tag{19}$$

$$I(d) = I_{\mathrm{in}} \cdot (1-R)^2 \cdot e^{-2\frac{z}{\delta}} = I_{\mathrm{in}} \cdot 0.593^2 \cdot 0.2932 = I_{\mathrm{in}} \cdot 10.3\%. \tag{20}$$

4 What was left out from the session about Fresnel formulae...

5 Gaussian beam propagation example: concentric laser resonator

λ = 633 nm, z_M = 300 mm, beam divergence is θ_d = 0.6 mrad. What is the beam radius (w_0) at the beam waist? What is the Rayleigh range (z_R)? What is the mirror radius necessary for stable laser operation (R)? We can disregard the beam-distortion effects of the mirror substrate.



$$\theta_d = \frac{\lambda}{\pi w_0}\ [\mathrm{rad}] \ \rightarrow \ w_0 = \frac{\lambda}{\pi\theta_\mathrm{d}} = \frac{633\cdot 10^{-6}\ \mathrm{mm}}{\pi 0.0006} = 0.336\ \mathrm{mm} \tag{21}$$

$$z_\mathrm{R} = \frac{\pi w_0^2}{\lambda} = 560.3\ \mathrm{mm} \tag{22}$$

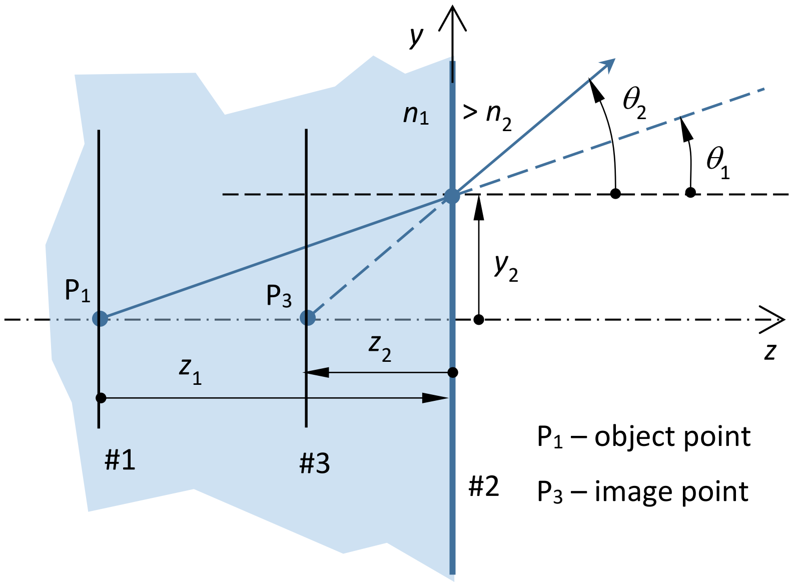
$$R(z) = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right] = \frac{z_M}{2}\left[1 + \left(\frac{2z_R}{z_M}\right)^2\right] =$$

$$= \frac{300}{2}\left[1 + \left(\frac{2 \cdot 560.3}{300}\right)^2\right] = 2243 \text{ mm.} \tag{23}$$

The radius obtained is much larger than the distance of the mirrors (150 mm) measured form beam waist, which is in correspondence with being well inside the Rayleigh range.

6 Example for refraction at planar surface: object point shifting

By the help of Snell's law and the matrix formalism let us describe light refraction at a plane surface for an object point being at a finite discance from it!



$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{24}$$

$$n_1\theta_1 \approx n_2\theta_2 \quad \rightarrow \quad n_1 \operatorname{tg}\theta_1 \approx n_2 \operatorname{tg}\theta_2 \tag{25}$$

$$n_1 \frac{y_2}{z_1} = n_2 \frac{y_2}{-z_2} \tag{26}$$

$$z_2 = -z_1 \frac{n_2}{n_1}. \tag{27}$$

Alternatively:

$$\mathbf{M} = \begin{bmatrix} 1 & \frac{z_2}{n_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{z_1}{n_1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{z_1}{n_1} + \frac{z_2}{n_2} \\ 0 & 1 \end{bmatrix} \tag{28}$$

There is imaging between planes 1 and 3, thus $B = 0$:

$$\frac{z_1}{n_1} + \frac{z_2}{n_2} = 0 \tag{29}$$

$$z_2 = -z_1 \frac{n_2}{n_1}. \tag{30}$$

That is we obtained the same result with both methods.