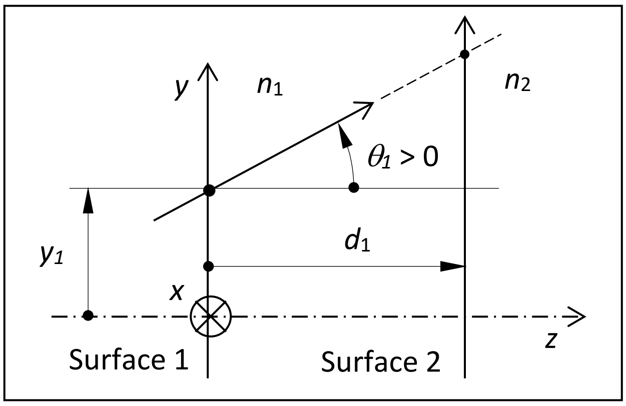
Physicist / Engineer physicist BSc, year III / semester 1
"Optics problem solving" – practice notes

PRACTICE 4

Dr. Gábor Erdei, 08-10-2025

1. Ray coordinates, optical direction cosines, signs



Optical direction cosines: *p* and *q*. E.g. $q \triangleq n \cdot \theta$

$$\left.\begin{aligned} y_2 &= A \cdot y_1 + B \cdot q_1 \\ q_2 &= C \cdot y_1 + D \cdot q_1 \end{aligned}\right\} \tag{1}$$

2. Transfer matrices, meaning of their elements

$$\begin{bmatrix} y_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ q_1 \end{bmatrix} \tag{2}$$

B – zero, if there is imaging between the two surfaces

A – lateral magnification (in case of imaging)

C – (−1) times the refractive power (system parameter)

D – *n*'/*n* times the angular magnification (in case of imaging)

Refractive power of a spherical refractive surface (surface number 2) – positive for focusing:

$$P_2 \triangleq \frac{n_2}{f_2} = \frac{n_2 - n_1}{r_2} \tag{3}$$

The radius is positive if the sphere center is placed in +*z* direction from the vertex.

3. Matrix of free-space propagation

For light propagation in a homogeneous medium between two surfaces, the direction cosine does not depend on position, thus *C* = 0. Besides, *q* does not change between the first and second surface ($q_1 = q_2$), so: *D* = 1. The position difference from one surface to the other in paraxial approximation:

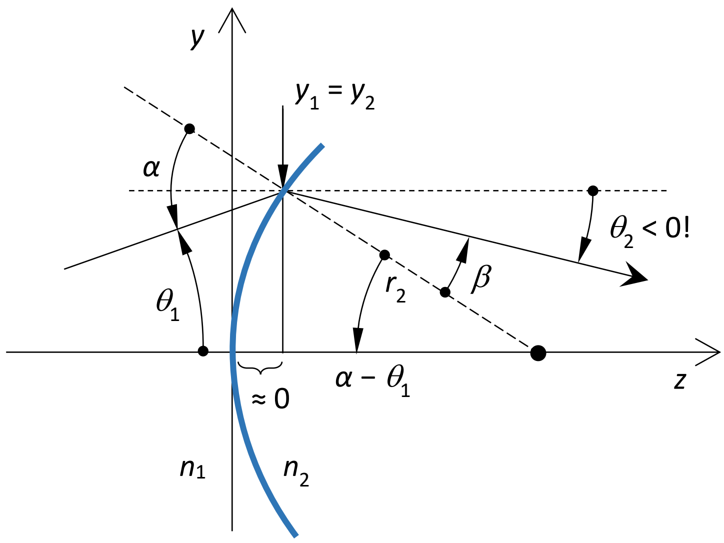
$$y_2 = y_1 + \theta_1 \cdot d_1 = 1 \cdot y_1 + \frac{q_1}{n_1} \cdot d_1 \tag{4}$$

Hence *A* = 1 and $B = d_1/n_1$, that is

$$\mathbf{T_1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & \frac{d_1}{n_1} \\ 0 & 1 \end{bmatrix}. \tag{5}$$

4. Matrix of refraction at a spherical surface, focal distance

At this problem we regard surfaces 1 and 2, between which propagation/refraction is examined, to be exactly *coincident*. Hence, at refraction the position of the ray intercept does not change ($y_1 = y_2$): A = 1, and it neither depends on the angle of incidence, so B = 0.



$$n_1\alpha = n_2\beta \tag{6}$$

$$\alpha - \theta_1 \approx \frac{y_1}{r_2} \tag{7}$$

$$(\alpha - \theta_1) = \beta - \theta_2 \;\; \rightarrow \;\; \theta_2 = -(\alpha - \theta_1) + \beta \tag{8}$$

$$\theta_2 = -\frac{y_1}{r_2} + \frac{n_1}{n_2}\alpha \;\; \rightarrow \;\; \theta_2 = -\frac{y_1}{r_2} + \frac{n_1}{n_2}\left(\frac{y_1}{r_2} + \theta_1\right) \tag{9}$$

$$\theta_2 = \frac{1}{r_2}\left(\frac{n_1}{n_2} - 1\right)y_1 + \frac{n_1}{n_2}\theta_1 \tag{10}$$

$$n_2\theta_2 = -\frac{n_2 - n_1}{r_2}y_1 + n_1\theta_1 \tag{11}$$

Thus C = $-P$ and D = 1, where P_2 is the refractive power of the surface. With this:

$$\mathbf{R}_2 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P_2 & 1 \end{bmatrix} \tag{12}$$

Generally, if the system includes several (e.g. N) subsequent refractions and free-space propagations, then the matrices are multiplied with each other:

$$\mathbf{M} = \mathbf{T}_N \cdot \mathbf{R}_N \cdot \ldots \cdot \mathbf{T}_2 \cdot \mathbf{R}_2 \cdot \mathbf{T}_1 \tag{13}$$

5. Spherical mirror

Expressing the matrix of a single reflective spherical surface of r_2 radius placed in a medium of n_1 refractive index. According to the law of reflection:

$$\beta = -\alpha \tag{14}$$

This is as if we had $n_2 = -n_1$ in the law of refracion, and the other equations remain unchanged:

$$P_2 = -2\frac{n_1}{r_2}. \tag{15}$$

Thus the matrix is of form (12) again.

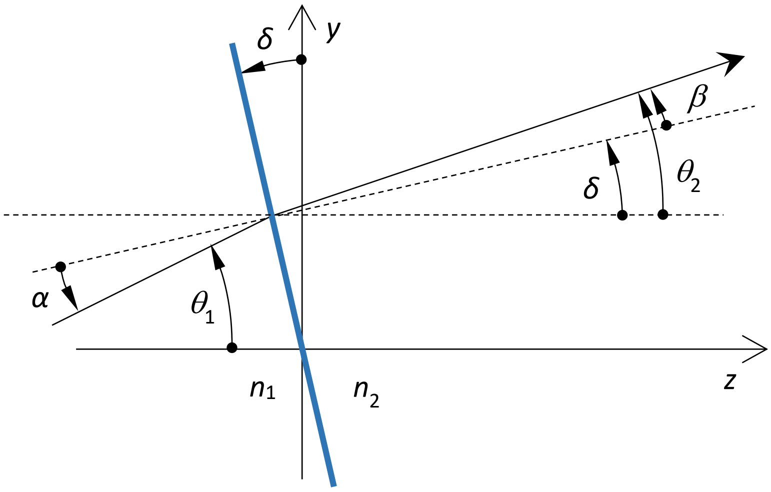
6. Tilted planar refractive surface

$$n_1\alpha = n_2\beta \tag{16}$$

$$\theta_1 = \alpha + \delta \tag{17}$$

$$\theta_2 = \beta + \delta \tag{18}$$

$$\theta_2 = \frac{n_1}{n_2}\alpha + \delta = \frac{n_1}{n_2}(\theta_1 - \delta) + \delta \tag{19}$$



$$n_2\theta_2 = n_1(\theta_1 - \delta) + n_2\delta = n_1\theta_1 + \delta(n_2 - n_1) \tag{20}$$

$$\begin{bmatrix} y_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ q_1 \end{bmatrix} + \begin{bmatrix} E \\ F \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ \delta(n_2 - n_1) \end{bmatrix} \tag{22}$$

Thus a tilted plane does not have a multiplicative matrix, but a constant additive vector instead. If $\delta = 0$, then we simply get a unit matrix.

7. Thin lens

Lens formula (s – object distance, s' – image dist., f' – effective focal length in image space):

$$\frac{n'}{s'} = \frac{n'}{f'} + \frac{n}{s} \tag{23}$$

The matrix of a thin lens from the first surface to the second:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P_1 - P_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix} \tag{24}$$

The matrix of a thin lens from the first surface to the focal plane:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & \frac{f'}{n'} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix} = \begin{bmatrix} 1 - P\frac{f'}{n'} & \frac{f'}{n'} \\ -P & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{f'}{n'} \\ -P & 1 \end{bmatrix} \tag{25}$$

Let us trace a light ray in parallel with the axis ($q_1 = 0$) through the system to the focal plane (this is surface 3) so that it intersects surface 1 at point y_1. In the focal plane:

$$\begin{bmatrix} y_3 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{f'}{n'} \\ -P & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} \frac{f'}{n'} q_1 \\ -Py_1 + q_1 \end{bmatrix} \tag{26}$$

In our case this is:

$$\begin{bmatrix} y_3 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -Py_1 \end{bmatrix}, \tag{27}$$

that is the position does not depend on the ray coordinate, and the direction is:

$$\theta_3 = \frac{q_3}{n'} = -\frac{Py_1}{n'} = -\frac{y_1}{f'}. \tag{28}$$

8. Thick lens

Matrix of a thick lens from the first surface to the second:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{d_1}{n_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P_1 & 1 \end{bmatrix} \tag{29}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P_2 & 1 \end{bmatrix} \begin{bmatrix} 1 - P_1\frac{d_1}{n_1} & \frac{d_1}{n_1} \\ -P_1 & 1 \end{bmatrix} \tag{30}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 - P_1\frac{d_1}{n_1} & \frac{d_1}{n_1} \\ -P_2\left(1 - P_1\frac{d_1}{n_1}\right) - P_1 & -P_2\frac{d_1}{n_1} + 1 \end{bmatrix} \tag{31}$$

$$P = P_2\left(1 - P_1\frac{d_1}{n_1}\right) + P_1 = P_1 + P_2 - P_1P_2\frac{d_1}{n_1}. \tag{32}$$

The above relationship is very useful considering that it can also be used for two thin lenses placed at d_1 distance from each other, since the formula is indifferent for what produces the P_1 and P_2 refractive powers: either a surface or a thin lens.

9. Complex systems, principal planes

Let the refractive power of a symmetric focusing lens surrounded by air ($n = n' = 1.0$) be $P = +4$ diopters, the lens thickness $d_1 = 3.0$ mm, the refractive index $n_1 = 1.5$. What is the focal length of the lens?

$$P \triangleq \frac{n'}{f'} = 4\ \mathrm{m}^{-1} = 0.004\ \mathrm{mm}^{-1} \tag{33}$$

From this $f' = 1/0.004 = 250$ mm. Considering the lens thickness too, how much is the radius of curvature ($r_1 = -r_2$)?

$$P_1 = P_2 = \frac{n_1 - \mathrm{n}}{r_1} = \frac{1.5 - 1}{r_1} = \frac{0.5}{r_1}. \tag{34}$$

$$P = P_1 + P_2 - P_1P_2\frac{d_1}{n_1} = \frac{1}{r_1} - \frac{0.25}{r_1^2}\frac{3}{1.5} = 0.004 \tag{35}$$

$$0.004\, r_1^2 - r_1 + 0.5 = 0 \tag{36}$$

From this $r_{1,2} = 0.501$ mm or $r_{1,2} = 249.5$ mm – the latter is the solution corresponding to ordinary lenses we are familiar with. The 0.501 mm radius makes sense too, this corresponds to the case when there is an intermediate focal point between the surfaces.

What is the refractive power of one surface? From (34): $P_1 = 2.004 \cdot 10^{-3}$ 1/mm. What are the elements of the matrix of the lens between its two surfaces? Based on (31):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0.996 & 2 \\ -0.004 & 0.996 \end{bmatrix} \tag{37}$$

Consequence of Liouville's theorem: the determinant of the transfer matrix is always 1.0. This also means that the matrix elements are not independent of each other. This relationship can be utilized to check the validity of our calculations. In our case the determinant value is: 1.000016.

Where are the principal planes of the lens located?

$$\mathbf{M} = \begin{bmatrix} 1 & \frac{z'}{n'} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & -\frac{z}{n} \\ 0 & 1 \end{bmatrix} \tag{38}$$

$$\mathbf{M} = \begin{bmatrix} 1 & \frac{z'}{n'} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & -A\frac{z}{n} + B \\ C & -C\frac{z}{n} + D \end{bmatrix} \tag{39}$$

$$\mathbf{M} = \begin{bmatrix} A + \frac{z'}{n'}C & -A\frac{z}{n} + B + \frac{z'}{n'}\left(-C\frac{z}{n} + D\right) \\ C & -C\frac{z}{n} + D \end{bmatrix} \tag{40}$$

There is imaging between the two principal planes, between which the magnification is by definition +1. Thus:

$$A + \frac{z'}{n'}C = 1 \;\; \Rightarrow \;\; z' = (1 - A)\frac{n'}{C} \tag{41}$$

From this z' = −1.00 mm. From the value of angular magnification we get:

$$-C\frac{z}{n} + D = 1 \;\; \Rightarrow \;\; z = (D - 1)\frac{n}{C}. \tag{42}$$

From this: z = 1.00 mm. Here we used the Lagrange-Helmholtz equation that is valid if there is imaging between the two surfaces (this is practically the law of energy conservation): $m \cdot m_\alpha = n/n'$, that is for the case of imaging $A \cdot D = 1$.