PRACTICE 2

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1 Application of the complex refractive index: analysis of a welding shield glass

λ_0 = 313 nm (UVB), T = 0.00002%, only internal absoprtion is considered (now we neglect surface reflectance), normal incidence, linear polarization. n = 1.5, how much is κ? Axis z points in the direction of the surface normal. Material thickness: z = 6.0 mm.

$$\tilde{n} \triangleq n - \mathrm{i} \cdot \kappa \tag{1}$$

$$E_\mathrm{x}(z) = E_0 \cdot e^{-ik_\mathrm{re}z} \cdot e^{-k_\mathrm{im}z} \tag{2}$$

Intensity attenuation is described by the Beer-Lambert law, i.e. the absolute square of (2):

$$I(z) = I_0 \cdot e^{-2k_\mathrm{im}z} \tag{3}$$

In case of normal distribution it is true that:

$$k_\mathrm{im} = k_0 \cdot \kappa \tag{4}$$

$$I(z) = I_0 \cdot e^{-2k_0\kappa z} = T \cdot I_0 \tag{5}$$

$$-2k_0\kappa z = \ln(T) \tag{6}$$

$$\kappa = -\frac{\ln(T)}{2k_0 z} = -\frac{\ln(T)}{4\pi z}\lambda_0 \tag{7}$$

From which $\kappa = 6.4 \cdot 10^{-5}$.

2 Application of the complex refractive index: penetration depth for metals

λ_0 = 550 nm, normal incidence, linear polarization. Aluminum: n = 1.0, κ = 6.6, how much is δ? Axis z points in the direction of the surface normal. Based on (5):

$$I(z) = I_0 \cdot e^{-2k_0\kappa z} = I_0 \cdot e^{-2\frac{z}{\delta}} \tag{8}$$

$$\delta = \frac{1}{k_\mathrm{im}} = \frac{1}{k_0\kappa} = \frac{\lambda_0}{2\pi\kappa} \tag{9}$$

From which δ = 13 nm.

3 Paraxial wave equation in dielectrics

We search for the electric field in a separable form (complex notation!):

$$\mathbf{E}(\mathbf{r}, t) \coloneqq \mathbf{E}(\mathbf{r}) \cdot e^{i\omega t} \tag{10}$$

Helmholtz equation:

$$\nabla^2\mathbf{E}(\mathbf{r}) + \varepsilon\mu\omega^2\mathbf{E}(\mathbf{r}) = 0 \;\; \rightarrow \;\; \nabla^2\mathbf{E}(\mathbf{r}) + k^2\mathbf{E}(\mathbf{r}) = 0 \;\text{ mivel }\; k = \frac{2\pi}{\lambda} = \frac{\omega}{\mathrm{v}} \tag{11}$$

E is complex, k is the length of the wave vector of a plane wave of ω angular frequency in the given medium, and λ is its wavelength ($\lambda = \lambda_0/n$). Let the beam be polarized in x direction:

$$\mathbf{E}(\mathbf{r}) \;\rightarrow\; E_\mathrm{x}(\mathbf{r}) \tag{12}$$

$$\frac{\partial^2 E_\mathrm{x}}{\partial x^2} + \frac{\partial^2 E_\mathrm{x}}{\partial y^2} + \frac{\partial^2 E_\mathrm{x}}{\partial z^2} + k^2 E_\mathrm{x} = 0, \tag{13}$$

and let the z axis be its typical propagation direction:

$$E_\mathrm{x}(\mathbf{r}) \coloneqq E_0 \cdot u(x, y, z) \cdot e^{-ikz}. \tag{14}$$

$$\frac{1}{E_0}\frac{\partial^2 E_x}{\partial z^2} = -k^2 u \cdot e^{-ikz} - 2ik\frac{\partial u}{\partial z}\cdot e^{-ikz} + \frac{\partial^2 u}{\partial z^2}\cdot e^{-ikz} \approx -k^2 u \cdot e^{-ikz} - 2ik\frac{\partial u}{\partial z}\cdot e^{-ikz} \tag{15}$$

If the below condition satisfies, we are in the paraxial approximation:

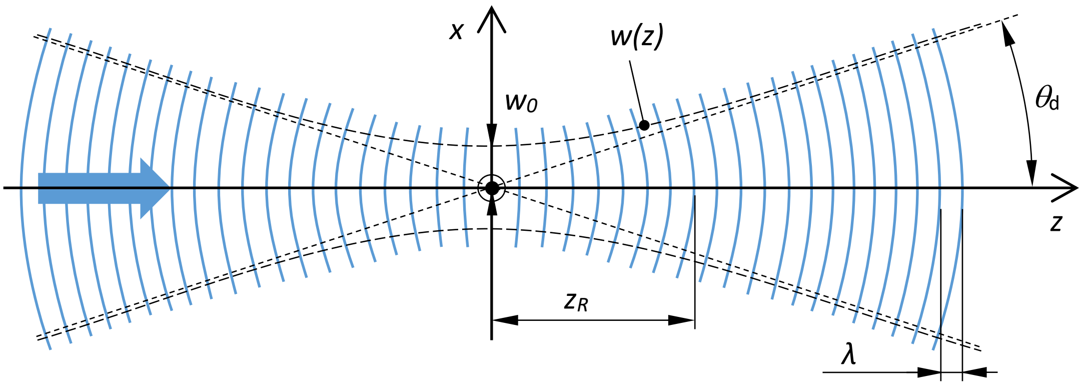
$$2k\left|\frac{\partial u}{\partial z}\right| \gg \left|\frac{\partial^2 u}{\partial z^2}\right| \quad \rightarrow \quad 4\pi\left|\frac{\partial u}{\partial z}\right|\lambda \gg \left|\frac{\partial^2 u}{\partial z^2}\right|\lambda^2 \tag{16}$$

where both sides were divided by k^2. Since $k^2 E_x$ cancels from (13), then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2ik\frac{\partial u}{\partial z} = 0. \tag{17}$$

This is the paraxial, time-invariant wave equation. Formally, it corresponds to the Schrödinger equation, if the potential is zero and $t \rightarrow z$.

4 Characteristics of a Gaussian beam



In case of free-space propagation the analytic solution to (17) is an integral equation: the Fresnel approximation of scalar diffraction, which will be discussed later. The Fresnel integral has eigenfunctions, the so-called Hermite-Gaussian beams, which also statisfy the above wave equation of course (proving this is much beyond the scope of a practice). The simplest form of these functions is the Gaussian beam, which is cylindrically symmetric to z:

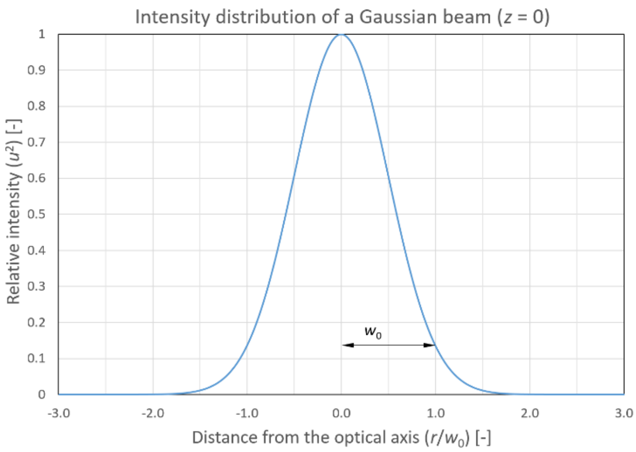
$$u(x,y,z) = \frac{w_0}{w}\cdot e^{-\left(\frac{r}{w}\right)^2}\cdot e^{-ik\frac{r^2}{2R}}\cdot e^{i\varphi} \tag{18}$$

where all the w, R, φ are quantities depending on z, and

$$r^2 \triangleq x^2 + y^2 \tag{19}$$

The spatial dependence of the beam radius (radius of $1/e^2$ intensity):

$$w^2(z) = w_0^2\left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right] \tag{20}$$



Let us imagine momentarily that we translate the reference frame to a positive *z* position, where the wavefront bends *to the left*. In paraxial approximation the wavefront of a spherical beam with its center on the *z* axis and a radius of $R < 0$ crossing this origo is (see last session):

$$z \approx \frac{r^2}{2R}, \quad \text{where} \quad r^2 = x^2 + y^2. \tag{21}$$

Since the sign of *R* is considered positive for Gaussian beams when the center of the circle is shifted in negative direction on the *z* axis, the equation of the left-bending wavefront is:

$$z \approx -\frac{r^2}{2R}. \tag{22}$$

Measuring the distance from the wavefront to the *x* axis multiplies this by -1. Since the phase of the field along the *x* axis is determined by the optical path from the wavefront to the *x* axis, which is a phase delay, we need to switch sign again, which explains the sign in (18).

Now back to the original origo. The radius of curvature of the wavefront depends on *z* as:

$$R(z) = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right]. \tag{23}$$

If *z* tends to ∞, then $R \rightarrow z$, i.e. the beam transforms into a spherical wave centered on the origo. The Gouy shift of the phase:

$$\varphi(z) = \mathrm{arc\,tan}\left(\frac{\lambda z}{\pi w_0^2}\right) \tag{24}$$

φ tends to π/2, if z → $+\infty$. The phase advances, i.e. it is like the local wavelength were a little longer than that of a plane wave! Near *z* = 0 in equation (24) we have $\mathrm{arc\,tan}(x) \approx x$, i.e. the phase varies about linearly. Then the relative wavelength increase is the following:

$$\varphi(z)|_{z=0} \approx \left.\frac{\partial\varphi}{\partial z}\right|_{z=0} \cdot z = \frac{\lambda z}{\pi w_0^2} \quad ; \quad e^{-ikz} \cdot e^{i\frac{\partial\varphi}{\partial z} \cdot z} = e^{-ik'z} \quad \rightarrow \quad \frac{\Delta\lambda}{\lambda} \approx \frac{\lambda^2}{2\pi^2 w_0^2} \tag{25}$$

E.g.: if $w_0 = 10\lambda$, then the wavelength increase is 5%. Here the other parameters in summary:

$$R(z)|_{z=0} = \infty \tag{26}$$

$$w(z)|_{z=0} = w_0 \tag{27}$$

$$E_\mathrm{x}(z)|_{z=0} = E_0 \tag{28}$$

The *z* = 0 position, where the beam radius is the smallest (w_0) is called the *beam waist*.

Laser resonators can only work in stability (i.e. when light does not scatters out from them by diffraction), if Hermite-Gaussian beams propagate in them – these are called transverse resonator modes. In this terminology the Gaussian beam is the so-called TEM_{00} mode.

5 Divergence of a Gaussian beam

Every Gaussian beam can be well approximated by a spherical beam when (see (20)):

$$\frac{\lambda z}{\pi w_0^2} \gg 1 \quad \rightarrow \quad z \gg \frac{\pi w_0^2}{\lambda} \tag{29}$$

i.e. if $z \rightarrow \infty$ (far-field approximation). Then the θ_d divergence:

$$w_\infty \triangleq \lim_{z\rightarrow\infty} w(z) = \frac{\lambda z}{\pi w_0} \quad \rightarrow \quad \theta_\mathrm{d} \triangleq \mathrm{arc\,tan}\left(\frac{w_\infty}{z}\right) \approx \frac{w_\infty}{z} = \frac{\lambda}{\pi w_0}\ [\mathrm{rad}]. \tag{30}$$

Beam parameter product:

$$BPP \triangleq \theta_\mathrm{d} \cdot w_0 = \frac{\lambda}{\pi} \tag{31}$$

It can be proved that among all possible beams tha value of *BPP* is the smallest for a Gaussian beam (cf. Heisenberg uncertainty relation).

How much is the divergence of a Gaussian beam (see He-Ne laser) of λ_0 = 633 nm wavelength, w_0 = 0.8 mm radius? Based on (30) $\theta_d = 633\cdot10^{-6}/0.8/\pi = 0.25$ mrad. What is the spot diameter at a distance of 1 km? $2\cdot0.25\cdot10^{-3}\cdot1000 = 0.5$ m. Can we really use the far-field approximation here? In order to answer the question first we interpret the expression of $\pi w_0^2/\lambda$ in (29).

6 Minimum radius of curvature wavefront of a Gaussian beam

The radius of curvature of the wavefront changes continuously with z. Where is R minimum?

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z+\frac{1}{z}\left(\frac{\pi w_0^2}{\lambda}\right)^2\right] = 1-\frac{1}{z^2}\left(\frac{\pi w_0^2}{\lambda}\right)^2 = 0 \tag{32}$$

This is called the Rayleigh range:

$$z_\mathrm{R} \triangleq \frac{\pi w_0^2}{\lambda} \tag{33}$$

What is the beam radius here?

$$w(z_\mathrm{R}) = w_0\sqrt{2} \tag{34}$$

What is the Rayleigh range of the previous He-Ne laser? $z_\mathrm{R} = \pi\cdot0.8^2/(633\cdot10^{-6}) = 3176$ mm, which is much smaller than 1 km, thus we can use the far-field approximation indeed.

7 Conditions of the paraxial approximation for a Gaussian beam – Supplementary

Let us examine condition (16) on the optical axis, i.e. r = 0. In this case (18) leaves us:

$$u(x,y,z) = \frac{w_0}{w}\cdot e^{i\varphi} \tag{35}$$

This expression has first and second derivatives too. Making these operations, (16) simplifies to the following form:

$$z+\frac{z_R^2}{z} \gg 4\lambda \tag{36}$$

The left handside has a minimum exactly when $z = z_\mathrm{R}$. Then

$$z_R \gg 2\lambda \tag{37}$$

Substituting (33) into this:

$$\frac{\pi w_0^2}{\lambda} \gg 2\lambda \quad\rightarrow\quad w_0 \gg \sqrt{\frac{2}{\pi}}\lambda = 0.80\lambda \tag{38}$$

The other condition of paraxial approximation is that E_z and E_y << E_x, i.e. the scalar approximation must be valid. Experience shows that we can disregard the effects of vector diffraction for beams with a numerical aperture (NA) smaller than 0.6:

$$NA \triangleq n\cdot\sin(\theta_d) < 0.6 \tag{39}$$

If n = 1 then:

$$\theta_d < 0.64\ \mathrm{rad} \tag{40}$$

Based on (30):

$$\frac{\lambda}{\pi w_0} < 0.64 \quad\rightarrow\quad w_0 > 0.50\lambda \tag{41}$$

From (38) and (40) the former is more constraining. In summary we can state that in case of analysing Gaussian beams the conditions of paraxial approximation are satisfied when

$$w_0 \geq 8\lambda. \tag{42}$$