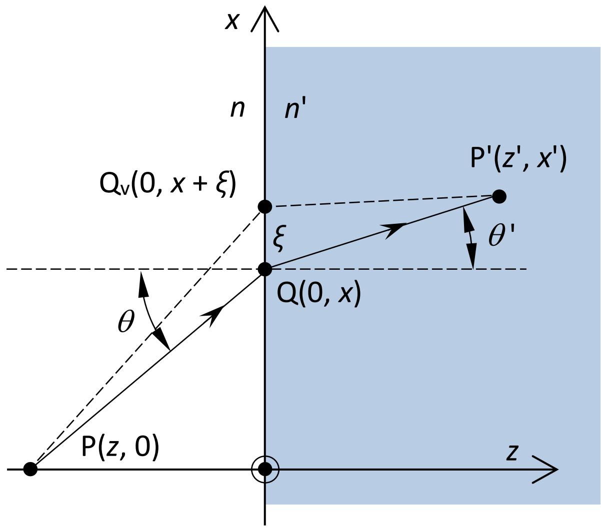
PRACTICE 1

Dr. Gábor Erdei, 11-09-2025

1. Demonstration of Fermat's principle: proving the law of refraction

We know that light propagates along straight lines in a homogeneous medium, thus we examine only such possible trajectories. By Q we denote that point where the light ray intersects the surface in reality. The task is to find the $\theta - \theta'$ pair for a plane wave refracting at a planar surface, where the optical path length is stationary at first order. The light ray representing the incident plane wave is launched from point P towards point P'.



$$OPL(\mathrm{PQ_vP'}) = n\sqrt{(x+\xi)^2+z^2} + n'\sqrt{(x'-x-\xi)^2+z'^2} \tag{1}$$

$$\frac{dOPL}{d\xi} = \frac{n}{2}\frac{2(x+\xi)}{\sqrt{(x+\xi)^2+z^2}} - \frac{n'}{2}\frac{2(x'-x-\xi)}{\sqrt{(x'-x-\xi)^2+z'^2}} = 0 \tag{2}$$

When $\xi = 0$ (indirect proof):

$$\frac{nx}{\sqrt{x^2+z^2}} = \frac{n'(x'-x)}{\sqrt{(x'-x)^2+z'^2}} \tag{3}$$

According to trigonometric relationships:

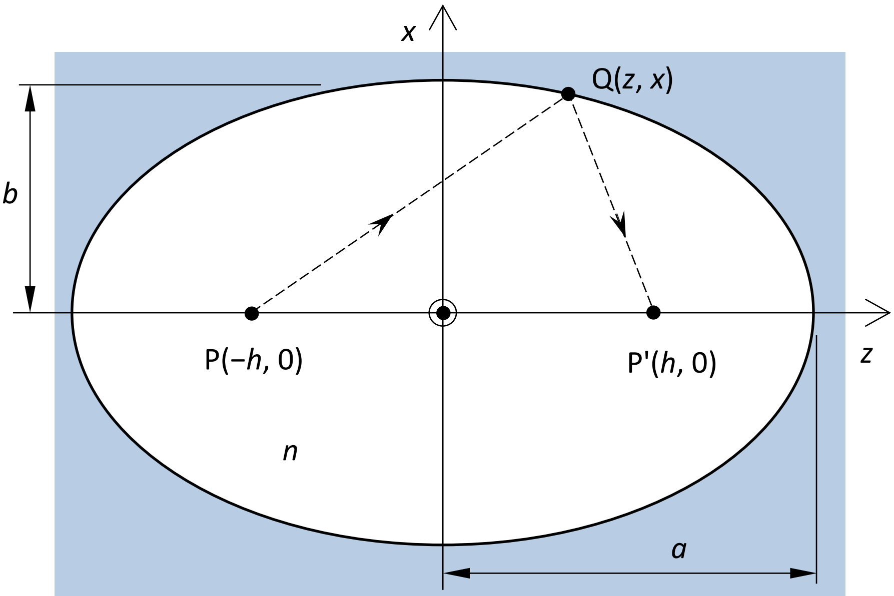
$$\sin(\theta) = \frac{x}{\sqrt{x^2+z^2}} \quad \text{and} \quad \sin(\theta') = \frac{x'-x}{\sqrt{(x'-x)^2+z'^2}} \tag{4}$$

Substituting these into (3):

$$n\cdot\sin(\theta) = n'\cdot\sin(\theta')\,. \tag{5}$$

2. Application of Fermat's principle: imaging with an ellipsoid mirror

Ellipsoid mirrors are known to perfectly image in geometric sense a point source being in one of its focal points (P) into its other focal point (P'). By the help of Fermat's principle let us prove this statement for points P and P' beind at equal distances from the *x* axis of symmetry. For this we search for the surface shape where *OPL* = const. for all light rays.



According to Fermat's principle the *OPL* is equal for all PQP' paths:

$$OPL(\mathrm{PQP'}) = n\sqrt{(z+\mathrm{h})^2 + x^2} + n\sqrt{(h-z)^2 + x^2}\,. \tag{6}$$

Since all paths are equal, we can easily determine the value of *OPL* by light propagation along the *z* axis:

$$OPL = n\big((a+h) + (a-h)\big) = n2a, \tag{7}$$

Thus from (6) we obtain (*n* cancels):

$$\sqrt{(z+h)^2 + x^2} + \sqrt{(h-z)^2 + x^2} = 2a\,, \tag{8}$$

Rearranging and squaring (8):

$$(h-z)^2 + x^2 = 4a^2 - 4a\sqrt{(z+h)^2 + x^2} + ((z+h)^2 + x^2), \tag{9}$$

$$a^2 + hz = a\sqrt{(z+h)^2 + x^2}\,, \tag{10}$$

$$a^4 + 2a^2hz + h^2z^2 = a^2(z+h)^2 + a^2x^2, \tag{11}$$

$$a^4 + 2a^2hz + h^2z^2 = a^2z^2 + 2a^2hz + a^2h^2 + a^2x^2, \tag{12}$$

$$a^4 + h^2z^2 = a^2z^2 + a^2h^2 + a^2x^2, \tag{13}$$

$$a^2 + \frac{h^2z^2}{a^2} = z^2 + h^2 + x^2, \tag{14}$$

Rearranging and grouping the terms:

$$a^2 - h^2 = z^2\frac{a^2 - h^2}{a^2} + x^2, \tag{15}$$

From this we can see, that we got the equation of an ellipse. In this case *a*, *b* and *h* quantities cannot be independent. For point Q being on axis *x* it is true that *x* = *b*:

$$OPL = n2\sqrt{h^2 + b^2} = n2a \;\;\rightarrow\;\; b^2 = a^2 - h^2 \tag{16}$$

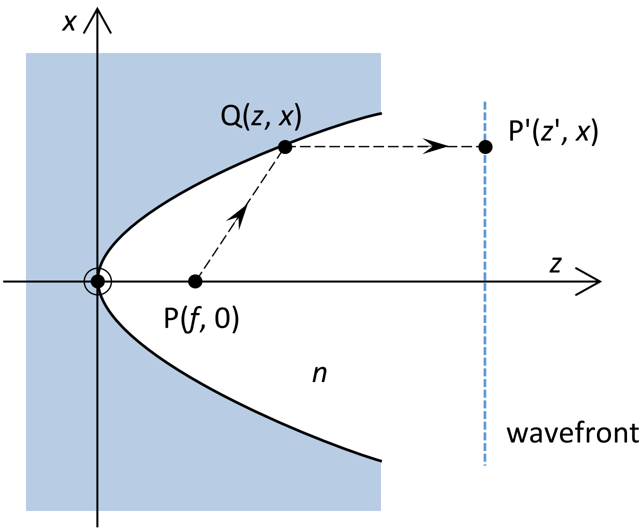
Based on (16) we get the equation of an ellipse indeed:

$$b^2 = z^2\frac{b^2}{a^2} + x^2, \tag{17}$$

$$\frac{z^2}{a^2} + \frac{x^2}{b^2} = 1. \tag{18}$$

3. Application of Fermat's principle: imaging with a paraboloid mirror

A paraboloid mirror images perfectly in geometrical sense a point source being on its optical axis at infinity into its focal point and vica versa. By the help of Fermat's principle let us prove this statement. Let the object point in the focal point (P). Since it is inconvenient to trace a ray to an image point at infinity, we will split the *OPL* calculation in two. Let us consider a wavefront that propagates towards the image point in the vicinity of the mirror. The image point is at infinity, thus this wavefront is a planar surface. From this plane to the image point (P') the optical path length is uniform for each light ray, thus it serves as a constant term in the *OPL* calculation, which we can disregard. It is sufficient thus to examine the optical path between the object point and this plane wavefront along the PQP' trajectory. Since the image point is at infinity, the QP' section is parallel to the *z* axis (optical axis).



$$OPL(\mathrm{PQP'}) = n\sqrt{(z-f)^2 + x^2} + n(z' - z), \tag{19}$$

For the sake of simplicity let $z' = f$ and $n = 1$. Then

$$OPL = \sqrt{(z-f)^2 + x^2} + (f - z), \tag{20}$$

If we examine the light ray along the *z* axis:

$$OPL = 2f. \tag{21}$$

Since all optical paths are equal:

$$\sqrt{(z-f)^2 + x^2} + (f - z) = 2f, \tag{22}$$

$$(z-f)^2 + x^2 = (f + z)^2, \tag{23}$$

$$z^2 - 2zf + f^2 + x^2 = f^2 + 2fz + z^2, \tag{24}$$

$$x^2 = 4fz, \tag{25}$$

$$z = \frac{x^2}{4f}. \tag{26}$$

4. Spherical mirror in paraxial approximation – additional thought

Based on the paraboloid mirror we can easily understand how spherical mirrors focus light, though they can only produce geometrically perfect imaging in the paraxial approximation (see later). Let us determine the focal length of a spherical mirror in the paraxial approximation (i.e. close to the optical axis). The surface of a sphere crossing the origo while having a radius *R* and center $z_0 = +R$ in the *x*-*z* plane:

$$x^2 + (z - R)^2 = R^2. \tag{27}$$

In paraxial approximation, i.e. when $x \ll R$:

$$z = R \pm \sqrt{R^2 - x^2} = R - R\sqrt{1 - \left(\frac{x}{R}\right)^2} \approx R - R\left(1 - \frac{1}{2}\left(\frac{x}{R}\right)^2\right), \tag{28}$$

Where we applied a Taylor series expansion. The sign of *R* is positive by definition when its center is shifted in positive direction along the *z* axis. Hence:

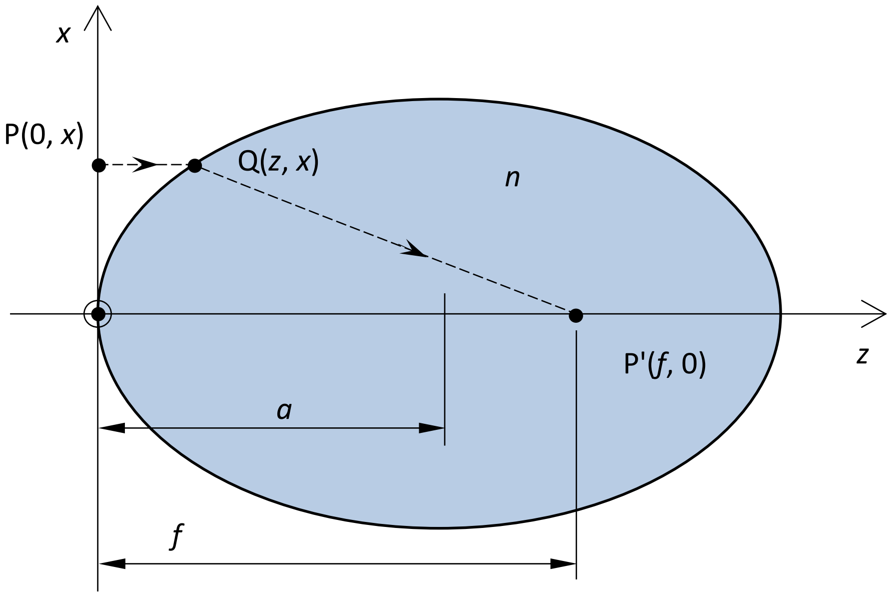
$$z \approx \frac{x^2}{2R}\,. \tag{29}$$

Comparing this with (26) we obtain the focal length of the paraboloid tangent to the sphere:

$$f = \frac{R}{2}. \tag{30}$$

5. Application of Fermat's principle: imaging with an ellipsoid lens

By the help of Fermat's principle let us determine what shape of cylindrically symmetric lens surface images an object point on its axis at infinity geometrically perfecty into the P' focal point. The *OPL* calculation is made again from a plane wavefront (P) of the object point, which is at the *z* = 0 position. The PQ segment is again parallel to the *z* axis (optical axis).



Let us examine the optical path with respect to the plane wavefront being in the *x*-*y* plane:

$$OPL(\mathrm{PQP'}) = z + n\sqrt{x^2 + (f - z)^2}. \tag{31}$$

On the optical axis:

$$OPL(\mathrm{PQP'}) = nf. \tag{32}$$

According to Fermat's principle all PQP' optical paths are equal:

$$z + n\sqrt{x^2 + (f - z)^2} = nf, \tag{33}$$

$$n^2(x^2 + (f - z)^2) = (nf - z)^2, \tag{34}$$

$$n^2x^2 + n^2f^2 - n^2 2fz + n^2z^2 = n^2f^2 - 2nfz + z^2, \tag{35}$$

Arranging all terms containing *x* to the left, and all terms containing *z* to the right:

$$n^2x^2 = (1 - n^2)z^2 + 2nfz(n - 1), \tag{36}$$

$$n^2x^2 = (1 - n^2)z^2 - 2nfz(1 - n). \tag{37}$$

Changing the right handside to a complete square expression:

$$n^2x^2 = (1 - n^2)\left(z - \frac{nf(1 - n)}{1 - n^2}\right)^2 - \frac{n^2f^2(1 - n)^2}{1 - n^2}, \tag{38}$$

$$n^2x^2 = (1 - n^2)\left(z - \frac{nf}{1 + n}\right)^2 - \frac{n^2f^2(1 - n)}{1 + n}, \tag{39}$$

$$\frac{n^2x^2}{1 - n^2} = \left(z - \frac{nf}{1 + n}\right)^2 - \frac{n^2f^2}{(1 + n)^2}, \tag{40}$$

Arranging to the left all terms containing *x*, *z*, and the constant to the rigth:

$$\frac{n^2x^2}{n^2 - 1} + \left(z - \frac{nf}{1 + n}\right)^2 = \frac{n^2f^2}{(1 + n)^2}. \tag{41}$$

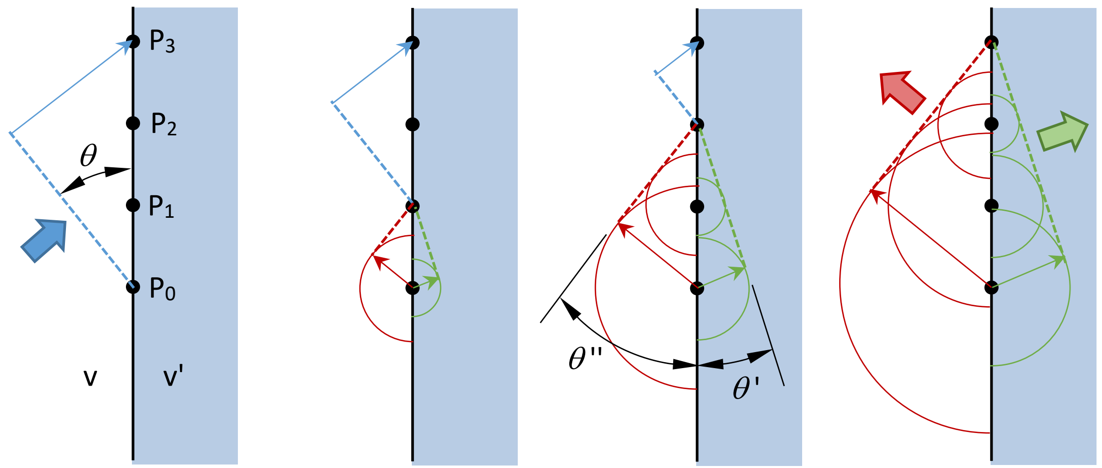
Dividing both sides with the right handside we get the equation of an ellipse:

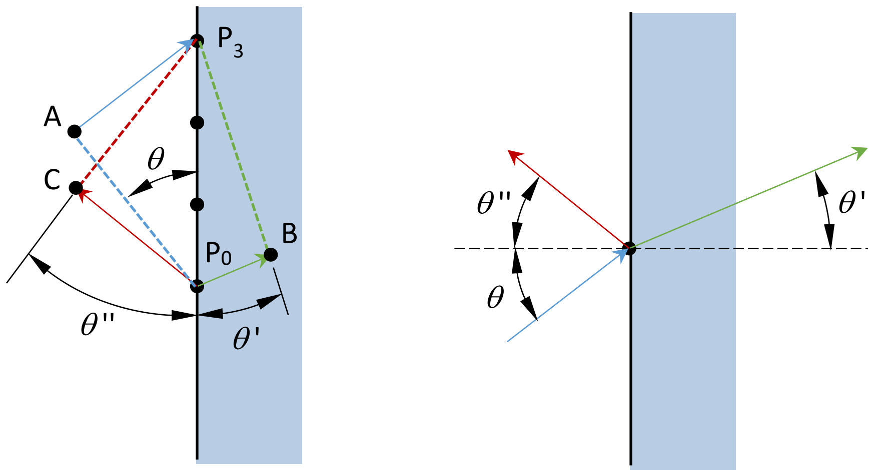
$$\frac{x^2}{\frac{n - 1}{n + 1}f^2} + \frac{\left(z - \frac{nf}{1 + n}\right)^2}{\frac{n^2f^2}{(1 + n)^2}} = 1 \quad \Leftrightarrow \quad \frac{x^2}{b^2} + \frac{(z - a)^2}{a^2} = 1. \tag{42}$$

From this the long half axis of the ellipse (*a*):

$$a = \frac{nf}{1 + n}. \tag{43}$$

6. Application of Huygens' principle: laws of light refraction and reflection – supplement





$$\mathrm{v}' < \mathrm{v} \;;\; \overline{\mathrm{AP_3}} = \mathrm{v}\cdot 3\Delta t \;;\; \overline{\mathrm{P_0B}} = \mathrm{v}'\cdot 3\Delta t\,;\; \overline{\mathrm{P_0C}} = \mathrm{v}\cdot 3\Delta t \tag{44}$$

$$\sin(\theta) = \frac{\overline{\mathrm{AP_3}}}{\overline{\mathrm{P_0P_3}}} \;;\; \sin(\theta') = \frac{\overline{\mathrm{P_0B}}}{\overline{\mathrm{P_0P_3}}} \;;\; \sin(\theta'') = \frac{\overline{\mathrm{P_0C}}}{\overline{\mathrm{P_0P_3}}} \tag{45}$$

$$\frac{\sin(\theta)}{\mathrm{v}\cdot 3\Delta t} = \frac{1}{\overline{\mathrm{P_0P_3}}} \;;\; \frac{\sin(\theta')}{\mathrm{v}'\cdot 3\Delta t} = \frac{1}{\overline{\mathrm{P_0P_3}}} \;;\; \frac{\sin(\theta'')}{\mathrm{v}\cdot 3\Delta t} = \frac{1}{\overline{\mathrm{P_0P_3}}} \tag{46}$$

$$\sin(\theta) = \sin(\theta'') \;;\; \frac{\sin(\theta)}{\mathrm{v}} = \frac{\sin(\theta')}{\mathrm{v}'} \tag{47}$$

If we introduce the $n = c/\mathrm{v}$ és $n' = c/\mathrm{v}'$ refractive indices:

$$\theta = \theta''\,;\; n\cdot\sin(\theta) = n'\cdot\sin(\theta')\,. \tag{48}$$