Relativity problem set 3.

1. Let us consider a line connecting the timelike separated points x and y. Let us regard this line as the world line of a particle propagating from x to y. Give the proper time parametrization of this line and verify that $U \cdot U = -1$!

2. In the situation of the twin paradox for Paul staying on earth the elapsed time is 50 years. On the other hand for Peter travelling with $\beta = 0.96c$ (c is the speed of light) the elapsed time for the trip is 14 years. This is OK because from the reference frame of Paul, Peter's clocks are proceeding slower. According to relativity from the reference frame of Peter, Paul's clocks are proceeding slower as well. However, when the twins meet again Paul is 71 and Peter is merely 35. Then it seems that the theory contains a basic logical inconsistency. What is the problem with the argument as presented above? Analyse the situation via drawing a spacetime diagram and solve the paradox!

3. Let us express the spacelike

$$A^b = \frac{d^2 x^b}{d\tau^2}$$

four-acceleration in terms of the three-acceleration and the three-velocity known from Newton's theory. Calculate the Minkowski length $A \cdot A$. Let us show that in the local inertial frame of the accelerating observer $A \cdot A = ||\mathbf{a}||^2$ where

$$\mathbf{a} = \frac{d^2\mathbf{x}}{(dx^4)^2} = \frac{1}{c^2}\frac{d^2\mathbf{x}}{dt^2}.$$

4. Let

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \qquad \mathrm{Det} S = 1.$$

The set of such matrices forms the group $SL(2, \mathbb{C})$. Let moreover

$$H = \begin{pmatrix} x^4 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^4 - x^3 \end{pmatrix} = H^\dagger = \overline{H}^T$$

be the matrix associated to the x^b ($b = 1, 2, 3, 4$) spacetime coordinates. For the transformed coordinates consider the matrix

$$\hat{H} = \begin{pmatrix} \hat{x}^4 + \hat{x}^3 & \hat{x}^1 - i\hat{x}^2 \\ \hat{x}^1 + i\hat{x}^2 & \hat{x}^4 - \hat{x}^3 \end{pmatrix}$$

where $\hat{x}^b$ is defined as follows

$$\hat{H} = SHS^\dagger.$$

Let us write the coordinates showing up in $\hat{H}$ in the form $\hat{x}^b = \Lambda(S)^b{}_c x^c$!

a. Let us show that $\Lambda(S)$ is the 4×4 matrix of an orthochronous Lorentz transformation!

b. Verify that the $S \mapsto \Lambda(S)$ map f is a group homomorphism where $f : SL(2, \mathbb{C}) \to \mathcal{L}_+^\uparrow$. (The group multiplication corresponds to ordinary matrix multiplication in the set of 2×2 or respectively 4×4 matrices.)

c. What is the kernel of f? What kind of relationship then we can establish between the groups $SL(2, \mathbb{C})$ and $\mathcal{L}_+^\uparrow$?

Hints: Let us show first that $\Lambda(S)$ is a Lorentz transformation, and next the orthochron property. Proving that the Lorentz transformtion is also proper (i.e. $\mathrm{Det}\Lambda = 1$) is technically more demanding. For part **c.** let us recall the isomorphism theorem of the group theory course.