Relativity problems 2.

1. Show that if we assume that for $\beta = \frac{v}{c}$, related to a newly defined parameter θ (rapidity) via $\beta = f(\theta)$, the

$$f(\theta_1 + \theta_2) = \frac{f(\theta_1) + f(\theta_2)}{1 + f(\theta_1)f(\theta_2)}, \qquad f(d\theta) = d\theta + \dots$$

constraint holds, then the UNIQUE solution of the constraint is $\beta = \tanh\theta$! The dots refer to higher order terms in $d\theta$. (Hint: plug in $\theta_1 = \theta, \theta_2 = d\theta$ and after Taylor expansion and keeping only terms up to first order in $d\theta$, try to obtain a differential equation for $f(\theta)$.)

2. Let us define the boost in the x^1 direction (defined by the hyperbolic rotation in the x^1x^4 plane) by the matrix

$$B_1(\theta) = \begin{pmatrix} \cosh\theta & 0 & 0 & -\sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\theta & 0 & 0 & \cosh\theta \end{pmatrix}, \qquad -\infty < \theta < \infty$$

Similarly one can define the boosts $B_2(\theta), B_3(\theta)$ representing hyperbolic rotations in the (x^2x^4) és (x^3x^4) planes. Let us also consider the rotations $R_1(\varphi), R_2(\varphi), R_3(\varphi)$, rotating in the x^2x^3, x^1x^3, x^1x^2 planes. For example for the rotation in the (x^1x^2) plane (rotation around the x^3 axis) we have

$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad 0 \le \varphi < 2\pi.$$

Let us calculate the infinitesimal generators of the three boosts and the three rotations! Let us denote the 4×4 matrices obtained during this calculation by $\mathcal{L}_j$ for the three rotations and by $\mathcal{K}_j, j = 1, 2, 3$ for the three boosts.

A: Calculate the commutation relations of the six generators $\mathcal{L}_i$ and $\mathcal{K}_j$ for $i, j = 1, 2, 3$!

B: For the rotations it is well-known that the infinitesimal generators of the rotations can also be realized by differential operators $L_j = -\sum_{k,l=1}^{3} \varepsilon_{jkl} x_k \partial_l$ acting on smooth functions of the three variables x^1, x^2, x^3. ($x_k \equiv x^k$ and $\partial_k = \frac{\partial}{\partial x^k}, j, k, l = 1, 2, 3$). How can we realize with differential operators the boosts, that are now acting on smooth functions of the four variables x^1, x^2, x^3, x^4? (Denote the corresponding differential operators by $K_j, j = 1, 2, 3$, and try to find combinations satisfying the commutation relations obtained in part **A**.)